On the equivalence of forward and backward induction reasoning under general preferences

Yi-Chun Chen*, Xuewen Qian

Department of Economics, National University of Singapore, Singapore 117570

This version: Sept 2016

Abstract

The notion of extensive form rationalizability (EFR) proposed by Pearce (1984) formalizes the idea of forward induction reasoning. We study EFR under rather general preferences, called constantly monotone preferences, which only assumes that a strategy assuring a higher constant payoff is preferred to the one assuring a lower constant payoff. In the class of generic perfect information games, we show that the behavioral implications of backward and forward induction are observationally indistinguishable. Specifically, in this class of games, EFR and backward induction yield the same terminal node under any preference model which admits subjective expect utility model; moreover, EFR procedure coincides with the backward iterated dominance procedure under the largest model which admits all constantly monotone preferences. *JEL Classification:* C70, C72.

Keywords: Extensive form rationalizability; general preferences; backward induction

*Corresponding author. *E-mail:* ecsycc@nus.edu.sg (Y. Chen), a0049466@u.nus.edu (X. Qian).

1 Introduction

The game-theoretic solution concept of rationalizability, proposed by Bernheim (1984) and Pearce (1984), is logical implication of common knowledge of rationality in the standard expected utility. Pearce (1984) also put forward the notion of extensive form rationalizability (EFR) as an extension to extensive games. Roughly speaking, EFR strategies are those surviving a process of iterated elimination procedure. In each step, all strategies that are "never sequentially-best responses" are eliminated. The procedure stops when no such strategy exists. Battigalli (1997) pointed out that EFR can be characterized as the sequential rationality to a hierarchies of beliefs systems which conform to the best rationalizable principle: Players' beliefs conditional upon observing a history must be consistent with the highest degree of "strategic sophistication" of their opponents. The best-rationalization principle is also related to the epistemic condition of "strong belief in rationality" (Battigalli and Siniscalchi, 2002). In this sense, EFR fundamentally differs from the notion of backward induction (BI) whose epistemic foundation is rationality and common belief of future rationality (Perea, 2014). Indeed, as the example in the next section suggests, EFR and BI might predict different strategies even in "generic"[1] perfect information (PI) games. However, as pointed out by Reny (1992), Battigalli (1997) and Heifetz and Perea (2015), in generic PI games, BI and EFR yield the same terminal node. Since EFR formalizes a fashion of forward induction reasoning: "A player should use all information she acquired about her opponents' past behavior in order to improve her prediction of their future, simultaneous, and past (unobserved) behavior, relying on the assumption that they are rational. (Battigalli, 1997, p.41)" This result is of fundamental significance. As an outside observer, one only observes the realization of the actual outcome or terminal node after the play of the game. The equivalence result, in terms of outcomes, implies that it is impossible to tell whether players' underlying reasoning procedures conform to forward induction or backward induction, since both yield exactly the same observation.[2]

Pearce's notion of EFR presumes Bayesian rationality: each player chooses a strategy that maximizes the subjective expected utility (SEU) axiomatized by Savage (1954). However, the Ellsberg Paradox and related experimental evidences cast a doubt on SEU model. Namely, decision makers usually have ambiguity aversion concerns which cannot be captured by a single prior distribution. Many generalizations of the SEU model are motivated then to relax Bayesian rationality. To name a few, for example, ordinal expected utility, probabilistically sophisticated preferences, Choquet expected utility theory and the multi-priors model. It is then meaningful to characterize EFR in a general preference model. Moreover, it is also interesting to examine whether the outcome equivalence result still holds in a broad sense. This paper thus aims to study these two questions.

Epstein and Wang (1996) offered a unified approach to study decision makers' rational strategic behavior under quite general preference setting (regular preference model). Their approach is applicable to study various notions of solution concepts in game theory. For

[1] Roughly speaking, "generic" means for each player, payoffs are all different among different terminal nodes.

[2] See also Heifetz and Perea (2015) for more discussions on this issue.

example, Epstein (1997) extended rationalizability to general preference models. Epstein and Wang's (1996) approach is virtually static and might not be directly applied to study the notion of EFR in the dynamic setting of extensive games. This paper then extends Epstein and Wang's (1996) approach and formulates "a model of conditional preference" which is applicable to EFR. Roughly speaking, each player holds a conditional preference system which specifies a conditional preference at each of his/her own information set. "A model of conditional preference" thus is a collection of admissible conditional preference systems of all players. The behavioral assumption embedded in the notion of conditional preference systems is the consequentialism. That is, conditioned on an information set h, strategic choices at information sets incompatible with h are irrelevant for the conditional preference at h.[3] Throughout this paper, only a rather weak condition, *constant monotonicity (CM)*, is imposed on the preference model. Roughly speaking, it only requires that a constant act a must be strictly preferred to another constant act b whenever $a > b$. In games situation, it implies that a strategy is strictly preferred to another strategy if the former strategy assures a constant payoff strictly higher than the latter one does. On the other hand, CM is silent on the preference order of strategies with payoff uncertainties. Note that CM preferences might not have utility representations and Epstein's (1997) regular preferences satisfy CM. Therefore, preference models considered in this paper is rather general.

One main result in this paper shows that in generic PI games, for any model of conditional preferences which admits all SEU preferences (consistent with Bayesian updating) and satisfies CM, the EFR strategy profiles yield the backward induction outcome (Theorem 1). That is, the outcome equivalence result in the literature remains true for a broad class of preference models. This result has rich implications. On the one hand, one may interpret it as an indistinguishable result. From an outside observer's point of view, neither strategy choices nor the preferences of players can be observed. The dynamic Ellsberg Paradox, viewed as a single-person game, shows that different preferences may yield different observable implications. However, in generic PI, EFR in a class of very general preference models yields the same set of outcomes of SEU preferences. The research line of indistinguishability/distinguishability appeals its significance in the literature: Bergemann and Morris (2009) applied "strategic distinguishability" to robust virtual implementation. On the other hand, the notion of EFR captures forward induction arguments which refine equilibrium concepts by restricting beliefs on off-equilibrium path (see Battigalli (1997); Battigalli and Siniscalchi (2002)). Theorem 1 implies that these refinements have no impact on behavior along equilibrium path for generic PI games, and the strategic implication of forward induction and backward induction coincides, i.e., backward induction reasoning and forward induction reasoning are also observationally indistinguishable. One may also see the backward induction outcome as a robust implication of EFR under general model of conditional preferences. Since CM condition is rather weak, the result covers all preference models discussed in the literature, e.g., the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the ordinal expected utility model, the lexicographic preference model, and the "strongly monotonic" preference model (see Chen

[3] The empirical study by Dominiak et al. (2012) show that more subjects act in line with consequentialism than with dynamic consistency and that this result is even stronger among ambiguity averse subjects.

and Luo, 2012). Theorem 1 can be regarded as a generalization to Battigalli's (1997) Theorem 4. Battigalli's (1997) proof, based on properties of Kohlberg and Mertens' (1986) "fully stable sets", can not be directly adapted to general preferences. We generalize the result by investigating two crutial properties of EFR. The first one regards the "dominance solvability" of EFR. We show that in generic PI games, EFR yields a unique terminal node under any preference model satifying CM. (See Lemma 1 in appendix.) That is, CM is sufficient to dominance solvability. This sufficient condition might be the weakest one since CM is the weakest requirement for rational preferences as far as we know. The second one says that if the preference model admits SEU model, then EFR outcomes under such model include the one under SEU model. (Lemma 2 in appendix.) This result is implied from the "outcome order independence" of EFR under SEU model shown by Chen and Micali (2013) and Luo et al. (2016).

Another main result shows that if the model admits all CM preferences, then the elimination procedure associated with EFR coincides with the backward iterated dominance procedure (Theorem 2). That is, under the "largest" model of CM preferences, the procedure to characterize EFR is exactly the one inspired by backward induction reasoning. As mentioned earlier, EFR and BI are conceptually different and have different implications of strategic choice for players. Nevertheless, this result shows that the conflict between those two solution concepts could be mitigated if we enlarge the underlying preference model.

The rest of this paper is organized as follows. In section 2, an illustrative example demonstrates the main results in this paper. Section 3 sets up the analytical framework. Section 4 presents the main theorems. Section 5 offers concluding remarks. To facilitate reading, all the proofs of are relegated to the Appendix.

2 An illustrative example

The following two-person centipede game demonstrates the main results in this paper for generic PI games (where generic means that no same payoff is assigned to two distinct terminal nodes for any player).

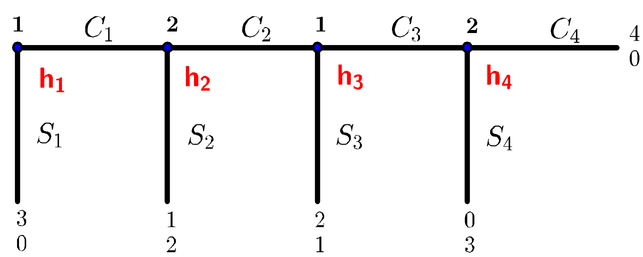

Fig. 1. A two-person game

Apparently, (S_1S_3, S_2S_4) is the unique BI strategy profile. Pearce's notion of EFR, or EFR with respect to SEU model, is the set of strategy profiles surviving iteratively eliminating never "sequential best replies" in the following way. In the first step, C_2C_4

is no better response than C_2S_4 conditioned on h_4; C_1S_3 is no better response than S_1S_3 conditioned on h_1 because by playing C_1S_3, player 1 either gets a payoff 1 or 2 while S_1S_3 secures the payoff 3. In the second step, player 2 considers that if h_2 is reached by a "rational" play, then it must be the case that player 1 chose C_1C_3. Therefore, C_2S_4 becomes player 2's only rational play. In the third step, C_1C_3 is no better response than S_1S_3 conditioned on h_1. Consequently, EFR set is $\{S_1S_3, S_1C_3\} \times \{C_2S_4\}$ which predicts the same terminal node (S_1) as the BI does. Note that the elimination procedure only involves strict dominance relation between pure strategies conditioned on reachable information sets. Thus EFR under the "strongly monotonic" preference model is exactly the same as that under SEU model.

Now consider a model which admits all CM preferences. Consequently, a strategy is strictly preferred to another strategy if both strategies "conditionally" assure constant payoffs and the former payoff is strictly higher than the latter one, and no specific preference ordering is primitively assumed once there is uncertainty involved (at the conditional). Particularly, in the first step, C_1S_3 could be an optimal reply to a CM preference which exhibits extremely ambiguity-seeking behavior. It is easy to see that under CM preference model, $C_2C_4; C_1C_3; C_1S_3$ are consecutively eliminated and left with the EFR set $\{S_1S_3, S_1C_3\} \times \{S_2S_4, S_2C_4\}$ whose outcome is the same as the BI outcome. Moreover, the elimination procedure exactly matches with the *backward iterated dominance procedure*: In each step, if backward induction deletes action a at node h, then delete all the strategies reaching h and choosing a (see, e.g., Osborne and Rubinstein, page 108).

Note that EFR set under SEU model is not a subset of that under CM model. It means that EFR is not monotone in preference model. The absence of monotonicity is related the "order dependent" issue of EFR. Suppose in each step along the EFR procedure, instead of eliminating all of the never "sequential best replies", we only eliminate some of them and stops when there is no more never "sequential best replies" left. This corresponds to a new elimination order. Different elimination orders may deliver different final sets. In particular, in the above example, the EFR procedure under CM model can be regarded as an alternative eliminition order of EFR under SEU model. Due to such ill behavior, the relationship between EFR under different preference model is unclear and thus it is difficult to find a straightforward proof for the indistinguishability result. However, EFR procedure under SEU model is "outcome order independent" in the sense that different eliminition order yields the same set of terminal nodes. This property provides us a short cut to relate EFR outcomes among different preference models. In fact, this property plays an important role in our proof.

3 Set-up

Consider a (finite) extensive form with perfect recall:[4]

$$\Gamma = (N, V, \{H_i\}_{i \in N}, \{A_h\}_{h \in \cup_{i \in N} H_i}),$$

where $N = \{1, 2, \cdots, n\}$ is the set of players, V is the set of nodes, H_i is the set of information sets for player $i \in N$, A_h is the set of actions available at information set h. Let $Z \subseteq V$ denote the set of terminal nodes. A payoff function for player i is a function $u_i : Z \to \mathbb{R}$. The game $\Gamma(u)$ is specified by the extensive form Γ and the payoffs $u \equiv (u_i)_{i \in N}$.[5]

A (pure) strategy of player i is a function that assigns an action in A_h to each information set $h \in H_i$. Let S_i denote the set of strategies of player i and $S \equiv \Pi_{i \in N} S_i \equiv S_i \times S_{-i}$ denote the set of strategy profiles. For a strategy profile $s \equiv (s_i)_{i \in N} \equiv (s_i, s_{-i}) \in S$, let $g(s)$ denote the corresponding terminal node realized.

Consider an information set h of player i. A pure strategy profile s reaches h if it reaches some node in h. The set of all profiles reaching h is denoted by $S(h)$, whose projections on S_i and S_{-i} are denoted by $S_i(h)$ and $S_{-i}(h)$ respectively. We say s_i (respectively s_{-i}) reaches h if $s_i \in S_i(h)$ (respectively $s_{-i} \in S_{-i}(h)$). By perfect recall, there is a unique sequence of actions of player i which leads to h, hence, $S(h) = S_i(h) \times S_{-i}(h)$.

From the decision theory point of view, each player is a decision maker who deals with opponents' strategic uncertainty and coordinates his/her sequential moves in the extensive game situation. The uncertainty partially resolves as the play of the game progresses. Specifically, once an information set $h \in H_i$ is realized and player i is about to move therein, he/she concludes that none of the strategy profiles which exclude h could be possibly played. That is, player i considers that h must be reachable by opponents' moves. Accordingly, player i restricts the opponents' strategic uncertainty to the set $S_{-i}(h)$. Each player is endowed with a *conditional preference system (cps)* to account for this process of uncertainty resolution. Formally, player i holds a cps $\succeq_i \equiv (\succeq_h)_{h \in H_i}$ such that for all $h \in H_i$, the complement of $S(h)$ is null in the sense of Savage (1954) for the conditional preference $\succeq_h$.[6] That is, payoffs on those terminal nodes incompatible with h are irrelevant for the conditional preference $\succeq_h$. One may refer to this formulation as a form of consequentialism. It embodies the idea that a preference conditional on an event should not depend on the consequences outside of that event. For more discussions, see, for example, Epstein and Le Breton (1993), Ghirardato (2002) and Hanany and Klibanoff (2007). Conditional preference systems generalize the conditional probability systems introduced by Renyi (1992).

Suppose player i holds a "working hypothesis" which states that strategy profiles being

[4] Since the formal description of an extensive form is by now standard (see, for instance, Kreps and Wilson (1982) and Osborne and Rubinstein (1994)), we here include the necessary notation only. We note that our approach in this paper can be easily extend to games with nature moves.

[5] To make the paper easy to read, we specify numerical payoffs to terminal nodes. However, it is not neccessary to do so. The analysis and main results in this paper apply as long as players are endowed with preference orderings over terminal nodes.

[6] $\forall s_i, s_i', t_i, t_i' \in S_i$, if $u_i(g(s_i, s_{-i})) = u_i(g(s_i', s_{-i}))$ and $u_i(g(t_i, s_{-i})) = u_i(g(t_i', s_{-i}))$ $\forall s_{-i} \in S_{-i}(h)$, then $s_i \succeq_h' t_i \Leftrightarrow s_i' \succeq_h t_i'$.

played are conformed to a subset $S' \subseteq S$. If the realization of $h \in H_i$ does not falsify this hypothesis, i.e., $S' \cap S(h) \neq \emptyset$, then at h, player i will naturally maintain the hypothesis which provides additional information. Otherwise, player i cannot maintain both consequentialism and the working hypothesis. Since consequentialism is primitively assumed in this paper, player i must abandon the latter in counterfactual cases. We say player i's cps $\succeq_i$ knows a subset $S' \subseteq S$ if he/she is sure of S' whenever S' is not falsified. Formally, we adopts the following definition:

Definition 1. Player i's cps $\succeq_i$ *knows* $S' \subseteq S$ if $\forall h \in H_i$, the complement of $S' \cap S(h)$ is Savage-null for $\succeq_h$whenever $S' \cap S(h) \neq \emptyset$.

This knowledge notion in this paper is in the same spirit of the "strong belief" operator in Battigalli and Siniscalchi (2002) and generalizes it to general preferences.

Consider a "*model of conditional preferences*" $\mathcal{P}(\Gamma(u)) \equiv \{\mathcal{P}_i(\cdot)\}_{i \in N}$ on $\Gamma(u)$, where $\mathcal{P}_i(\cdot)$ is defined for any subset in product form $S' = \times_{i \in N} S'_i \subseteq S$, and $\mathcal{P}_i(S')$ is interpreted as i's admissible cps which know S'.[7] Therefore, for every collection of cps in $\{\mathcal{P}_i(S')\}_{i \in N}$, the "reduced game" S' serves as a common "working hypothesis".

Definition 2. The *constantly monotone model* $\mathcal{P}^{CM}(\cdot)$ is defined as follows. For every $S' = \times_{i \in N} S'_i \subseteq S$, a cps $(\succeq_h)_{h \in H_i} \in \mathcal{P}_i^{CM}(S')$ if and only if it knows S' and for all $h \in H_i$, it satisfies *constant monotonicity (CM)*: $\forall s_i, s'_i \in S_i$,

$$[\forall s_{-i} \in S_{-i}(h), \, u_i(g(s_i, s_{-i})) = r > r' = u_i(g(s'_i, s_{-i}))] \Rightarrow [s_i \succ_h s'_i].$$

That is, CM requires that a strategy is strictly preferred to another strategy if the former strategy assures a constant payoff strictly higher than the latter one does. The restriction takes effect only under the case without any payoff uncertainty. Note that a CM preference is rather weak and might not have a utility representation. Any preference which violates CM would be considered as either trivial or irrational. In this sense, CM is self-evident and therefore no essential behavioral assumption, except consequentialism, is imposed on preferences throughout this paper. The following examples demonstrate that the analytical framework in this paper can be applied to games where the players have different kinds of preferences, including the standard SEU model (with Bayesian updating) and regular preference model as special cases. It is also applicable to model strategic behavior such as interactive ambiguity in extensive games.[8]

Example 1. The *SEU model*, denoted as $\mathcal{P}^{SEU}(\cdot)$, is defined through SEU representation based on conditional probability systems (see Renyi (1992)). For every $S' = \times_{i \in N} S'_i \subseteq S$,

[7] Since strategic implications are mainly concerns in this paper, for simplicity, we adopt a simple version of preference model here. We note that we can start from the state space and extend Epstein and Wang's (1996) way to construct a "model of conditional preference" from the primitive state space in dynamic setting. An advantage of the definition used in this paper is that it permits sharp results that can be interpreted as reflecting exclusively the more liberal meaning for the behavior in various game situations. (see Chen et al. (2016)) Note that, throughout this paper, no utility representation is assumed.

[8] See Ahn (2007) and Kajii (2005) for more discussions.

say $\mu \equiv (\mu_h)_{h \in H_i}$ is a conditional probability system over S'_{-i} if conditions (i)-(iii) hold for all $h, h' \in H_i$:

(i) μ_h is a probability distribution on $S_{-i}(h)$;

(ii) $\mu_h \left(S'_{-i} \cap S_{-i}(h) \right) = 1$ whenever $S'_{-i} \cap S_{-i}(h) \neq \emptyset$;

(iii) if $S''_{-i} \subseteq S_{-i}(h') \subseteq S_{-i}(h)$, then $\mu_h \left(S''_{-i} \right) = \mu_{h'} \left(S''_{-i} \right) \mu_h \left(S_{-i}(h') \right)$.[9]

A cps $(\succeq_h)_{h \in H_i} \in \mathcal{P}_i^{SEU}(S')$ if and only if for all $h \in H_i$, $\succeq_h$has an SEU representation as

$$u_i(s_i, \mu_h) = \textstyle\sum_{s_{-i} \in S'_{-i}} u_i \left(g\left(s_i, s_{-i} \right) \right) \mu_h \left(s_{-i} \right), \forall s_i \in S_i,$$

for some conditional probability system μ over S'_{-i}.

Example 2. The *regular preference model*, denoted as $\mathcal{P}^{Reg}(\cdot)$, is defined as follows. Conditioned on every $h \in H_i$, each strategy s_i can be identified with an act $\mathbf{s}_i(\cdot)|_h$ on $S_{-i}(h)$, i.e., $\forall s_{-i} \in S_{-i}(h)$, $\mathbf{s}_i(s_{-i})|_h = u_i(g(s_i, s_{-i}))$. For every $S' = \times_{i \in N} S'_i \subseteq S$, a cps $(\succeq_h)_{h \in H_i} \in \mathcal{P}_i^{Reg}(S')$ if and only if it knows S' and for all $h \in H_i$, there exists a regular preference $\succcurlyeq_h$on $S_{-i}(h)$ (see Epstein 1997, pp.6-7) such that $s_i \succeq_h s'_i \Leftrightarrow \mathbf{s}_i(\cdot)|_h \succcurlyeq_h \mathbf{s}'_i(\cdot)|_h$, $\forall s_i, s'_i \in S_i$.

Both SEU model and regular preference model satisfy CM and thus are submodels of $\mathcal{P}^{CM}(\cdot)$.

4 EFR and indistinguishability

Bayesian rationality is the usual behavioral assumption made in the literature; that is, each player forms a prior probability distribution over opponents' play and chooses a strategy to maximize the corresponding expected utility. In the dynamic counterpart, a Bayesian-rational player is assumed to choose a strategy which is a sequential best reply with respect to some conditional probability systems. The model of conditional preference in this paper considerably relaxes the behavioral assumption and might accommodate dynamic Ellsberg paradox. The following definition is the "sequential rationality" condition adopted in this paper for an arbitrary model of conditional preference.

Definition 3. Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$ and $S' = \times_{i \in N} S'_i \subseteq S$, $s_i \in S_i$ is a $\mathcal{P}$-*best reply* on S' if there exists a cps $\succeq_i \in \mathcal{P}_i(S')$ such that the following condition holds: $\forall h \in H_i$ reached by s_i, $s_i \succeq_h s'_i$ for all $s'_i \in S_i(h) \cap S'_i$.

That is, given a hypothetical "reduced game" S', s_i is $\mathcal{P}$-best reply on S' if it can be supported by some $\succeq_i$, which knows S', in the following sense: s_i must be the most preferred strategy conditioned on every information set not precluded by itself, compared to all the

[9]Condition (iii) says that if the information set h' follows h, then $\mu_{h'}$ is updated from μ_h by Bayes rule.

other strategies in the reduced game which reach that information set. Denote $r_i(\mathcal{P}_i(S'))$ as the set of all $\mathcal{P}$-best replies on S' for player i and denote $r(\mathcal{P}(S')) \equiv \times_{i\in N} r_i(\mathcal{P}_i(S'))$. The following definition extends Pearce's notion of EFR to arbitrary model of conditional preferences.

Definition 4. Given $\langle \Gamma(u), \mathcal{P}(\cdot)\rangle$, let $S^0 = S$. Define $S^1, ..., S^{n+1}$ inductively as $S^{n+1} = S^n \cap r(\mathcal{P}(S^n))$, then $S^\infty = \cap_{n\geq 0} S^n$ is the set of $\mathcal{P}$-extensive form rationalizable ($\mathcal{P}$-*EFR*) strategy profiles.

That is, after iteratively eliminating never $\mathcal{P}$-best replies, the set left is $\mathcal{P}$-*EFR*. $\mathcal{P}^{SEU}$-*EFR* characterizes EFR in Pearce (1984).

Following Battigalli (1997), say $\Gamma(u)$ is without relevant ties if $\forall i \in N$, $\forall s_i, s_i' \in S_i$ and $\forall s_{-i} \in S_{-i}$,

$$g(s_i, s_{-i}) \neq g(s_i', s_{-i}) \Rightarrow u_i(g(s_i, s_{-i})) \neq u_i(g(s_i', s_{-i})).$$

Theorem 1. *Given $\langle \Gamma(u), \mathcal{P}(\cdot)\rangle$, if $\Gamma(u)$ is a perfect information game without relevant ties and $\mathcal{P}^{SEU}(\cdot) \subseteq \mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, then the outcome of $\mathcal{P}$-EFR strategies ($g(\mathcal{P}$-EFR$)$) is unique, which is the same as the backward induction outcome.*

Theorem 1 generalizes Battigalli's (1997) Theorem 4 whose proof uses some properties of Kohlberg and Mertens' (1986) "fully stable sets". The proof for Theorem 1 in this paper relies on two results. The first one states that for any model of conditional preference $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, the set of $\mathcal{P}$-*EFR* strategy profiles reaches a unique terminal node. (See Lemma 1 in appendix.) The second one is the "outcome order independence" of $\mathcal{P}^{SEU}$-*EFR*, i.e., all iterative elimination orders of never $\mathcal{P}^{SEU}$-best replies reaches the same set of terminal nodes, a result shown by Chen and Micali (2013). $\mathcal{P}$-*EFR* is a special elimination order in the sense that all never $\mathcal{P}$-best replies are eliminated in each step. An arbitrary elimination order would be eliminating some of the never $\mathcal{P}$-best replies in each step and it stops when there is no never $\mathcal{P}$-best reply left. (See Definition and Lemma 2 in appendix.)

Remark. Accroding to proofs in the appendix, Theorem 1 can be further strengthened to arbitrary elimination orders. Formally speaking: Given $\langle \Gamma(u), \mathcal{P}(\cdot)\rangle$, if $\Gamma(u)$ is a PI game without relevant ties and $\mathcal{P}^{SEU}(\cdot) \subseteq \mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, then for an aribrary elimination order of never $\mathcal{P}$-best replies, the survived set yields the unique backward induction outcome.

Theorem 1 can be interpreted as an indistinguishable result: EFR strategic behavior under a class of very general preference models is observationally indistinguishable from that under the SEU model. Furthermore, base on the observed play of the game, one can not distinguish players using EFR reasoning process from those using reasoning process of backward induction. Note that without relevant ties is a generic condition, the indistinguishability thus generically holds for PI games. Chen and Luo (2012) showed an indistinguishable result for compact Hausdoff strategic game under "concave like" condition. It is worth to note that the result in this paper is topological free. Lo (2000) provided an indistinguishability result to all the models of preference that satisfy Savage's axiom P3, which is a form of monotonicity. The CM condition in this paper is even weaker and covers almost all preference models discussed in the literature.

Theorem 2. *Given $\langle \Gamma(u), \mathcal{P}^{CM}(\cdot) \rangle$, if $\Gamma(u)$ is a perfect information game without ties*[10]*, then $\mathcal{P}^{CM}$-EFR is the same as the set survives backward iterated dominance procedure.*

As demonstrated in the illustrative example in section 2, S_1S_3 is "sequential rational" if and only if S_1C_3 is because both yield the outcome S_1 regardless of the strategic choice of player 2. Therefore, EFR is conceptually not possible to deliver the BI strategy profile. However, Theorem 2 draws a connection between $\mathcal{P}^{CM}$-*EFR* and the backward iterated dominance procedure. That is, under $\mathcal{P}^{CM}(\cdot)$, the procedure to compute EFR is exactly the one inspired by BI reasoning. In this sense, enlarging the preference model to $\mathcal{P}^{CM}(\cdot)$ could partially mitigate the conflict between EFR and BI.

5 Non-genericity

For a non-generic PI game, the indistinguishability might fail. Consider the three-person PI game depicted in the following game tree.[11]

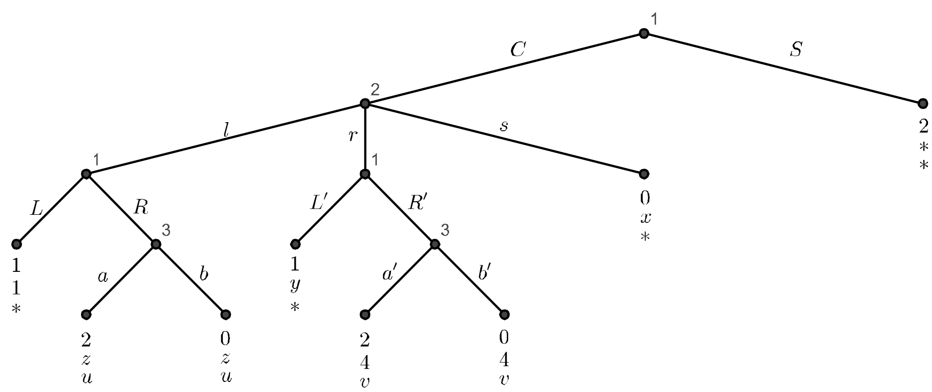

Fig. 2. A three-person game $\Gamma(x, y, z)$, where $x, y, z \in \mathbb{R}$

Player 3 can be interpreted as nature moves which generate payoff uncertainties for player 1 but not player 2. Consider 4 different preference models: SEU, ordinal expeted utility model (OEU), multi-prior (MP) and strongly monotone (Mon) preference model. By playing CLL', player 1's payoff is either 0 or 1, both are less than 2, the payoff guaranteed by playing S in the first place. Therefore, any strongly monotone preference can not support CLL'. It can be checked that for all 4 models, in the fisrt step, EFR only eliminates CLL' regardless the values of x, y, z. In the second step, player 2's strategy choices and payoff uncertainties can be summarized in the following table.

[10] if $\forall z, z' \in Z, \forall i \in N$, if $z \neq z'$, then $u_i(z) \neq u_i(z')$

[11] Payoffs marked by $*$ are irrelevant to solutions and thus unspecified.

	CLR'	CRL'	CRR'
s	x	x	x
l	1	z	z
r	4	y	4

Different combinations of (x, y, z) might deliver different strategic implications for player 2 in the second step. Finally, the following table describes non-rationalizable outcomes under different models and games.

	SEU	OEU	MP	Mon
$\Gamma(2,1,4)$	$\backslash Cs$			
$\Gamma(1,1,4)$	$\backslash Cs$	$\backslash Cs$		
$\Gamma(1,0,0)$	$\backslash ClL$	$\backslash ClL$	$\backslash ClL$	

The result shows that any pair of preference models could be distinguished in some game. Note that this game is non-generic and not dominance solvable. Consequently, the mutiplicity of rationalizable outcomes lead to distinguishability. Hence, this example shows dominance solvability is crutial to the indistinguishability.

6 Concluding Remarks

In this paper, we have formulated a model of conditional preferences and applied it to analyze the solution concept of EFR in extensive games. The main result show that behavioral implications of EFR are observationally indistinguishable among all preference models which admit SEU and satisfy constant monotonicity in generic PI games, and the EFR outcome is further indistinguishable from the BI outcome. Through out this paper, we impose a rather weak condition (probably the weakest), constant monotonicity, on preference models. All regular preferences satisfy this condition and our result is applicable for many preference models discussed in the literature, e.g., the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the ordinal expected utility model and the lexicographic preference model.

Unlike other indistinguishable results in literatures mentioned above, our result does not rely on any topological or algebraic structure. Instead, our result is based on the idea of consequentialism embedded in the conditional preferences which are rich enough to regulate rational behavior. In this respect, our result is sharp and sheds light on important and fundamental issues on rational strategic behavior in dynamic context.

7 Appendix: Proofs

Definition. Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$, a decreasing sequence of product sets $\{D^k\}_{k\geq 0}$ is an *elimination order of never $\mathcal{P}$-best replies* (*EON-$\mathcal{P}$*) if the following conditions hold: (i)

$D^0 = S$, (ii) $\forall k \geq 0, \forall s_i \in D_i^k \backslash D_i^{k+1}$, $s_i \notin r_i(\mathcal{P}_i(D^k))$, (iii) $D^\infty \subset r(\mathcal{P}(D^\infty))$ where $D^\infty \equiv \cap_{k\geq 0} D^k$.

Lemma 1. Given $\langle \Gamma(u), \mathcal{P}(\cdot)\rangle$, if $\Gamma(u)$ is a PI game without relevant ties and $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, then for an arbitrary *EON-$\mathcal{P}$*, $\{D^k\}_{k\geq 0}$, there is at most one outcome in D^∞ i.e., $|g(D^\infty)| \leq 1$. ($|\cdot|$ denotes the cardinality)

Proof of Lemma 1. Let $\mathcal{Y} = \{h \in \cup_i H_i : |g(D^\infty \cap S(h))| > 1\}$. Suppose on the contrary $|g(D^\infty)| > 1$, then $\mathcal{Y}$ includes the initial history and thus is nonempty. Pick $\underline{h} \in \mathcal{Y}$ such that it has no successor in $\mathcal{Y}$. $|g'(D^\infty \cap S(\underline{h}))| > 1 \Rightarrow \exists s, s' \in D^\infty \cap S(\underline{h})$ such that $g(s) \neq g(s')$. Without loss of generality, assume $\underline{h} \in H_i$. Then $\forall x_{-i} \in D_{-i}^\infty \cap S_{-i}(\underline{h})$, $g(s_i, x_{-i}) = g(s)$. Otherwise, since D^∞ is a product set, we can find a successor of $\underline{h}$ in $\mathcal{Y}$. Similarly, we have $\forall x_{-i} \in D_{-i}^\infty \cap S_{-i}(\underline{h})$, $g(s_i', x_{-i}) = g(s')$. Since the game is without relevant ties, $u_i(g(s)) \neq u_i(g(s'))$. Without loss of generality, assume $u_i(g(s')) > u_i(g(s))$. $\forall \succeq_i \in \mathcal{P}_i(D^\infty) \subseteq \mathcal{P}_i^{CM}(D^\infty)$, since $D^\infty \cap S(\underline{h}) \neq \emptyset$, the complement of $D^\infty \cap S(\underline{h})$ is Savage-null for $\succeq_{\underline{h}}$. By constant monotonicity, $s_i' \succ_{\underline{h}} s_i$. Therefore $s_i \notin \mathcal{P}_i(D^\infty)$, which contradicts $D^\infty \subseteq r(\mathcal{P}(D^\infty))$. □

Lemma 2. Given $\langle \Gamma(u), \mathcal{P}^{SEU}(\cdot)\rangle$, Let $\{D^k\}_{k\geq 0}$ and $\{\bar{D}^k\}_{k\geq 0}$ be elimination orders of never $\mathcal{P}^{SEU}$-best replies, then $g(D^\infty) = g(\bar{D}^\infty)$.

Proof of Lemma 2. According to Shimoji and Watson (1998), $\{D^k\}_{k\geq 0}$ and $\{\bar{D}^k\}_{k\geq 0}$ are elimination orders of conditional dominated strategies. According to Chen and Micali (2013), the elimination is order independent, i.e., $g(D^\infty) = g(\bar{D}^\infty)$. □

Proof of Theorem 1. Let $\{BI^k\}_{k\geq 0}$ be the backward iterated dominance procedure. It is easy to see that $\{BI^k\}_{k\geq 0}$ is an elimination order of never $\mathcal{P}^{SEU}$-best replies and $g(BI^\infty)$ is the unique backward induction outcome. Consider an arbitrary elimination order of never $\mathcal{P}$-best replies, $\{D^0, ..., D^K\}$. $\forall k \geq K$, define $D^{k+1} = D^k \cap r\left(\mathcal{P}^{SEU}\left(D^k\right)\right)$. Since $\mathcal{P}^{SEU} \subseteq \mathcal{P}$, $\{D^k\}_{k\geq 0}$ is an elimination order of never $\mathcal{P}^{SEU}$-best replies. By Lemma 2, $g(BI^\infty) = g(D^\infty) \subseteq g(D^K)$. Since $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, by Lemma 1, $\left|g(D^K)\right| \leq 1$. Thus, $g(D^K) = g(BI^\infty)$. The result follows since *$\mathcal{P}$-EFR* is the result of a special elimination order of never $\mathcal{P}$-best replies. □

Proof of Theorem 2. Let $\{BI^k\}_{k\geq 0}$ be the backward iterated dominance procedure. Define $\left\{S^k\right\}_{k\geq 0}$ inductively as $S^0 = S$ and $S^{k+1} = S^k \cap r\left(\mathcal{P}^{CM}\left(S^k\right)\right)$ for all $k \geq 0$. It suffices to show $BI^k = S^k$ for all $k \geq 0$. The equivalence is trivial when $k = 0$, and suppose it holds for $0, 1, ..., k$. It suffices to show (i) $s \in BI^k \backslash BI^{k+1} \Rightarrow s \in S^k \backslash S^{k+1}$; (ii) $s \in B^{k+1} \Rightarrow s \in S^{k+1}$. Let H^1 denote the set of last decision nodes. For all $l \geq 1$, inductively define H^{l+1} as the set of last decision nodes in $(\cup_{i\in N} H_i) \backslash H^l$. Denote a_h^* as the action prescribed by the backward induction at h. By definition,

$$BI^k = \left\{s \in S : \text{if } s \text{ reaches some } h \in \cup_{l=1}^k H^l, \text{ then } s_h = a_h^*\right\}$$

(i) $\forall s \in BI^k \backslash BI^{k+1}$, s reaches some $h^* \in H^{k+1}$ and $s_{h^*} \neq a_{h^*}^*$. Without loss of generality, assume $h^* \in H_i$. $\forall x \in BI^k \cap S(h^*)$, $x_h = a_h^*$ for all h which follows h^*. Therefore,

(s_i, x_{-i}) reaches the same terminal nodes for all $x \in BI^k \cap S(h^*)$. Let $s^* = (s_{-h^*}, a^*_{h^*})$, then $s^*_i \in BI^k \cap S(h^*)$. Similarly, (s^*_i, x_{-i}) reaches the same terminal node for all $x \in BI^k \cap S(h^*)$. Since $\Gamma(u)$ has no ties, by backward induction, $s^*_i \succeq_{h^*} s_i$ for all $\succeq_i \in \mathcal{P}^{CM}(BI^k)$. Thus $s \in r(S^k)$. By induction hypothesis, $s \in S^k \backslash S^{k+1}$.
(ii) $\forall s \in BI^{k+1}$. $\forall h \in \cup_{l=1}^{k+1} H^l$, if s reaches h, then $s_{h'} = a^*_{h'}$ for all h' equal to h or follows h, therefore, s is the most preference action for any constantly monotone $\succeq_h$. $\forall h \notin \cup_{l=1}^{k+1} H^l$, either A_h is a singleton or there are strategy profiles in $BI^k \cap S(h)$ leads to different terminal nodes. In the first case, s is the most preference action for any constantly monotone $\succeq_h$. In the second case, there exist $\succeq_h$which is certain of $BI^k \cap S(h)$ and supports the non-constant act $\mathbf{s}(\cdot)|_h$ as the most preferred one. Overall, $s \in r(B^k)$. By induction hypothesis, $s \in S^{k+1}$. □

References

1. D. Ahn, Hierarchies of ambiguous beliefs, JET 136(2007), 286-201.

2. P. Battigalli, On rationalizability in extensive games, JET 74(1997), 40-61.

3. Battigalli, Pierpaolo and Marciano Siniscalchi, "Strong belief and forward induction reasoning." JET 106(2002), 356-391.

4. Bergemann, D., Morris, S., Robust virtual implementation, TE 4(2009), 45-88.

5. Bernheim, B.D., Rationalizable strategic behavior, Econometrica 52(1984), 1007-1028.

6. Chen, J., Micali, S., The order independence of iterated dominance in extensive games, TE 8(2013), 125-163.

7. Dominiak, A., P. Durschz, and J.-P. Lefort, "A Dynamic Ellsberg Urn Experiment", GEB 75(2012), 625-638.

8. Epstein, L. G. and M. L. Breton, Dynamically consistent beliefs must be bayesian, JET 61(1993), 1-22.

9. Epstein, L.: Preference, rationalizability and equilibrium, JET 73(1997), 1-29.

10. Epstein, L., Wang, T., "beliefs about beliefs without probabilities", Econometrica 64(1996), 1343-1373.

11. Ghirardato, Paolo, "Revisiting Savage in a conditional world." Economic Theory 20(2002), 83-92.

12. E. Hanany, P. Klibanoff, Updating Preferences with Multiple Priors, TE 2(2007), 261-298.

13. A. Kajii, T. Ui, Incomplete Information Games with Multiple Priors, Japanese Economic Review 56(2005), 332-351.

14. E. Kohlberg and J.-F. Mertens, On the strategic stability of equilibria, Econometrica 54(1986), 1003-1039.

15. Lo, K.C., Rationalizability and the savage axioms, Economic Theory 15(2000), 727-733.

16. Luo, X., Qian, X. and C. Qu, "Unified approach to iterated elimination procedures in strategic games", Working paper 2016.

17. D. Pearce, Rationalizable strategic behavior and the problem of perfection, Econometrica 52(1984), 1029-1050.

18. P. Reny, Backward induction, normal-form perfection and explicable equilibria, Econometrica 60(1992), 626–649.

19. Savage, L., The Foundations of Statistics. NY: Wiley (1954)

20. M. Shimoji, J. Watson, Conditional Dominance, Rationalizability, and Game Forms, JET 83(1998), 161-195.