Generic equivalence between perfectly and sequentially rational strategic behavior*

Xiao Luo[a]†, Xuewen Qian[a], Yang Sun[a]

[a] *Department of Economics, National University of Singapore, Singapore 117570*

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Abstract

We follow Blume and Zame (Econometrica 62:783-794, 1994) to study the relationship between perfectly and sequentially rational strategic behavior from the point of view of semi-algebraic geometry. We present a unified framework for analyzing rational strategic behavior, with different structures of beliefs, in extensive games. In this paper, we show a general "generic" equivalence theorem between perfect rationality and sequential rationality, which is applicable to various solution concepts such as equilibrium, rationalizability, iterated dominance and MACA. *JEL Classification:* C70, C72.

Keywords: Extensive forms; generic payoffs; perfect rationality; sequential rationality; semi-algebraic sets

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†Corresponding author. Fax:+65 6775 2646. *E-mail:* ecslx@nus.edu.sg (X. Luo), a0049466@u.nus.edu (X. Qian), a0109707@u.nus.edu (Y. Sun).

1 Introduction

In dealing with imperfection in (finite) extensive games, Selten (1975) introduced the notion of (trembling-hand) perfect equilibrium. A perfect equilibrium is an equilibrium that takes the possibility of off-the-equilibrium play into account by assuming that the players, through the idea of "trembling hand", may choose all unintended strategies, albeit with small probabilities. In the spirit of Selten's (1975) perfectness, Kreps and Wilson (1982) proposed an alternative notion of sequential equilibrium, by imposing the so-called "sequential consistency" and "sequential rationality" on the behavior of every player. Sequential equilibrium is more inclusive and weaker than perfect equilibrium: every perfect equilibrium must be sequential. Kreps and Wilson (1982, Section 7) pointed out that the two concepts lead to similar prescriptions for equilibrium play: For each particular game form and for almost all assignments of payoffs to the terminal nodes, almost all sequential equilibria are perfect equilibria, and the sets of sequential and perfect equilibria fail to coincide only at payoffs where the perfect equilibrium correspondence fails to be upper hemi-continuous. Blume and Zame (1994) (hereafter BZ94) strengthened Kreps and Wilson's (1982) result and showed that: For almost all assignments of payoffs to the terminal nodes, the sets of sequential and perfect equilibria are identical. The research line of genericity in game theory sheds light on important and fundamental issues on rational strategic behavior; e.g., Kreps and Wilson (1982) and Kohlberg and Mertens (1986) applied Sard's Theorem and the Regular Value Theorem in differential topology to study equilibrium distributions over terminal nodes and the generic finiteness of equilibria components (see also Govindan and Wilson (2001, 2006, 2012), Govindan and McLennan (2001), Hillas and Kohlberg (2002), Haller and Lagunoff (2002), McKelvey and McLennan (1996), and Pimienta and Shen (2014) for more discussions).

BZ94 obtained the "generic" equivalence result by exploiting a special semi-algebraic structure of the graphs of the perfect and sequential equilibrium correspondences, because graphs of the two correspondences can each be written as a subset of a Euclidean space defined by a finite number of polynomial equalities and inequalities. As they pointed out,[1]

> "We believe that, just as differential topology has proved to be the right tool for studying the fine structure of the Walrasian equilibrium correspondence, so will real algebraic geometry prove to be the right tool for studying the fine structure of game-theoretic equilibrium correspondences. (BZ94, p.784)"

In this paper, we follow BZ94 to study the relationship between perfectly and sequentially rational strategic behavior in a broad sense, including equilibrium and non-equilibrium solution concepts, from the point of view of semi-algebraic geometry. We establish a general "generic" equivalence theorem between perfect rationality and sequential rationality in (finite) extensive games (Theorem 1). More specifically, we show that the difference between

[1] van Damme (1992, Theorem 2.6.1) presented an "almost all" theorem: In "almost all" normal form games, Nash equilibria are "regular" equilibria (hence proper equilibria). Nevertheless, as van Damme (1992, p.45) pointed out, the analysis "is of limited value for the study of extensive form games as any nontrivial such game gives rise to a nongeneric normal form."

the perfectly and sequentially rational correspondences under very feasible behavioral assumptions occurs only for "nongeneric" payoffs (which are included in a lower-dimensional semi-algebraic payoffs set). We also apply our general "generic" equivalence theorem to various solution concepts such as equilibrium, rationalizability, iterated dominance and MACA (Greenberg et al. (2009)); in particular, we obtain a variety of generic equivalence results as corollaries of Theorem 1 (Corollaries 1-4).

In a special class of "generic" games with perfect information (i.e., it is not a "nongeneric" case where, for some player, a same payoff is assigned to two distinct terminal nodes), it is fairly easy to see that perfect/sequential equilibrium yields the unique backward induction outcome (in terms of strategy profiles). In other words, sequential and perfect equilibria are generically identical in games with perfect information. The similar result indeed holds true for the notion of perfect/sequential rationalizability. That is, in the class of "generic" games with perfect information, both perfect/sequential equilibrium and rationalizability lead to the unique backward induction outcome, excluding a lower-dimensional set of payoffs (see Example in Section 2).

In this paper, we provide a unified approach to the "generic" relationship between perfectly and sequentially rational strategic behavior. We present a general framework to accommodate different structures of beliefs for different solution concepts and distinct players in games by restrictions on the scope of trembling sequences (specified by sets $\mathfrak{X}$). The graphs of perfectly and sequentially rational correspondences are related respectively to the closure and vertical closure of a set $\mathcal{R}^{\mathfrak{X}}$ of "perfectly-rational states" (Proposition 1). Based upon Generic Local Triviality in semi-algebraic geometry, we show that the closure and vertical closure of a semi-algebraic set almost coincide (Proposition 2). Consequently, perfectly and sequentially rational correspondences under the "structures of beliefs" $\mathfrak{X}$ are generically identical (Theorem 1). Our approach of this paper is rather feasible and applicable to various solution concepts, *as long as belief structures $\mathfrak{X}$ are semi-algebraic*. In this paper, we show that the belief structures behind many solution concepts in the literature are indeed semi-algebraic; for instance, if $\mathfrak{X}$ is restricted to a "common" trembling sequence for all players, Theorem 1 delivers BZ94's "generic" equivalence result for perfect and sequential equilibria.

One major feature of this paper is that, unlike BZ94, our approach does not rely directly on semi-algebraic properties of specific solution concepts. More specifically, BZ94's approach relies on the semi-algebraic property of sets of perfect/sequential equilibria, which are defined by polynomial equalities and inequalities, in finite dimensional Euclidean spaces. However, it is less clear that other kinds of perfect/sequential solution concepts – such as the sets of perfect/sequential rationalizable strategies – are semi-algebraic. Rather than working directly on the semi-algebraic property of solution concepts, we here take a different approach by exploiting the semi-algebraic property of the primitive set of "perfectly-rational states", which delivers a more general and fundamental generic equivalence between sequential and perfect rational behavior (Theorem 1). Moreover, BZ94 defined perfect and sequential equilibria by using "perturbed games" possibly with payoff perturbations (see Kreps and Wilson (1982)); our definitions in this paper are based on an alternative idea of "trembling strategies" possibly with payoff perturbations, so that our approach is feasible and applicable to unified solution concepts of "perfect-MACA" and "sequential-MACA" suggested by Greenberg et al. (2009) in complex situations. As a matter of fact, our paper provides an alternative approach to the study of the "generic" relationship between perfect and sequential equilibria in BZ94 (cf. also Section 5 for more discussions). From a technical perspective, BZ94 showed the generic equivalence result by using the generic continuity property of semi-algebraic correspondence; our proof is direct and dependent on the fact that the closure and vertical closure of a semi-algebraic set are generically identical (Proposition 2).

The rest of the paper is organized as follows. In Section 2, we provide an illustrative example to explain the general generic equivalence relationship between perfectly and sequentially rational strategic behavior. In Section 3, we present an analytical framework. In Section 4, we show a general "generic" equivalence theorem. We also obtain equivalence results for various solution concepts, as corollaries of the general generic equivalence theorem. Section 5 concludes. To facilitate reading, all the proofs are relegated to Appendix.

2 An Illustrative Example

The following two-person game demonstrates that there is a general relationship of "generic" equivalence between perfectly and sequentially rational strategic behavior (where a "generic" case means that no same payoff is assigned to distinct terminal nodes for each player).[2]

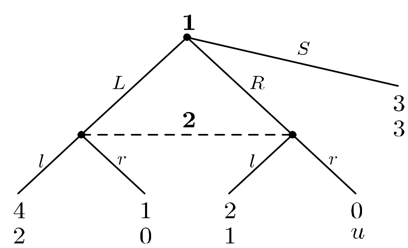

Fig. 1. A two-person game $\Gamma(u)$ *where* $u \in \mathbb{R}$

Apparently, L dominates R (for player 1); l dominates r (for player 2) if payoffs $u \leq 1$. It is easy to see that sequential equilibrium differs from perfect equilibrium only at "nongeneric" payoff $u = 1$. Moreover, the difference between perfect and sequential equilibria occurs only for "nongeneric" payoff(s) that are resided in a lower-dimensional payoffs space. BZ94 showed that: For "almost all" or "generic" assignments of payoffs to the terminal nodes, the sets of sequential and perfect equilibria are identical. (This example shows that there is no "generic" equivalence relationship between Myerson's (1978) proper equilibrium and perfect equilibrium: for "generic" payoffs $u > 1$, (S, r) is a perfect equilibrium but not a proper equilibrium.)

[2]More precisely, a statement is "generically" true if it is false only for a lower dimensional subset of the payoff vector space.

This sort of "generic" equivalence relationship indeed holds true for perfectly and sequentially rational strategic behavior: that is, "sequential rationality" differs from "perfect rationality" only at "nongeneric" payoff $u = 1$. For simplicity, we restrict attention to player 2's behavior in the game in Figure 1. Clearly, strategy r is not perfectly rational for player 2 since l (weakly) dominates r at "nongeneric" payoff $u = 1$. But, r is sequentially rational when $u = 1$ if player 2 holds a belief assessment $(p, 1-p) = (0, 1)$ at his information set; this belief assessment can be generated by a "trembling sequence" $x^\varepsilon \equiv \varepsilon^2 L + \varepsilon R + (1 - \varepsilon - \varepsilon^2)\, S$ as $\varepsilon \to 0$. Note that, although r is not optimal along the "trembling sequence" x^ε, it can be optimal by a slight perturbation on payoff u. (For instance, r is optimal along the "trembling sequence" x^ε under perturbed payoff $u^\varepsilon = 1 + 2\varepsilon$.) In other words, r can be perfectly rational under payoff perturbations. Subsequently, sequentially rational strategy r can be obtained from a limit point of "perfectly-rational states" $(x^\varepsilon, u^\varepsilon, r)$, i.e., $\lim_{\varepsilon \to 0} (x^\varepsilon, u^\varepsilon, r) = (1 \circ S, 1, r)$.

In fact, every sequentially rational strategy can be characterized by a limit point of perfectly-rational states (see Lemma 1 in Appendix), while every perfectly rational strategy is naturally associated with a limit point of perfectly-rational states, *without payoff perturbations*. That is, the set of sequentially (resp. perfectly) rational strategies can be characterized by the closure (resp. vertical closure) of the set of perfectly-rational states (see Proposition 1 in Section 3). By Generic Local Triviality in semi-algebraic geometry, the closure and vertical closure of the set of perfectly rational states are almost the same (see Proposition 2 in Section 4). Consequently, we obtain our central result of this paper: sequential rationality differs from perfect rationality only at "nongeneric" payoffs (see Theorem 1 in Section 4). This result is applicable to various kinds of solution concepts discussed in the literature, such as equilibrium, rationalizability, iterated dominance and MACA (see Corollaries 1-4 in Section 4). For example, if the belief structure allows different players to have distinct trembling sequences, our Theorem 1 yields a "generic" equivalence result for perfect and sequential rationalizability (Corollary 1); if the "structure of beliefs" is restricted to a "common" trembling sequence for all players, our Theorem 1 delivers BZ94's "generic" equivalence result for perfect and sequential equilibria (Corollary 2).

3 An analytical framework

3.1 Set-up

We consider a (finite) extensive form with perfect recall:[3]

$$\Gamma = (N, V, H, \{A_h\}_{h \in H}),$$

[3]Since the formal description of an extensive form is by now standard (see, for instance, Kreps and Wilson (1982) and Osborne and Rubinstein (1994)), we here include the necessary notation only. We note that our approach in this paper can be easily extend to games with nature moves.

where $N = \{1, 2, \cdots, n\}$ is the set of players, V is the set of nodes, H is the set of information sets, A_h is the set of actions available at information set h. Let $Z \subseteq V$ denote the set of terminal nodes. A payoff function for player i is a function $u_i : Z \to \mathbb{R}$. Let $U = \Pi_{i\in N} U_i$ where $U_i = \mathbb{R}^Z$ is the space of player i's payoff functions. The game $\Gamma(u)$ is specified by the extensive form Γ and the payoffs $u \in U$.

A mixed action at information set h is a probability distribution over the actions in A_h. Let $\mathbb{Y}_h$ denote the set of mixed actions at h (i.e. $\mathbb{Y}_h = \Delta(A_h)$). The set of player i's (behavior) strategies is $\mathbb{Y}_i = \Pi_{h\in H_i}\mathbb{Y}_h$ (where H_i is the set of player i's information sets). Let $\mathbb{Y} = \Pi_{i\in N}\mathbb{Y}_i$ and $\mathbb{Y}_{-i} = \Pi_{j\neq i}\ \mathbb{Y}_j$. (For a profile $y \in \mathbb{Y}$, we also write $y = (y_i, y_{-i}) = (y_h, y_{-h})$.)

The sets $\mathbb{Y}$, $\mathbb{Y}_i$, $\mathbb{Y}_{-i}$ and $\mathbb{Y}_{-h}$ can be viewed as semi-algebraic sets, which are defined by linear equalities and inequalities, in finite dimensional Euclidean spaces.[4] Fix a terminal node z, the probability $\Pr(z|y)$ that z is reached (from the initial node) is a polynomial function of $y \in \mathbb{Y}$. In game $\Gamma(u)$, i's expected payoff from $y \in Y$ is defined as: $v_i(y, u_i) = \Sigma_{z\in Z} u_i(z) \Pr\{z|y\}$, which is semi-algebraic on $\mathbb{Y} \times U_i$.

3.2 Perfect rationality and sequential rationality

Consider a game $\Gamma(u)$. For a strategy-profile vector $\mathbf{x} \in \mathbb{Y}^n$, we write $\mathbf{x} \equiv ({}^i x)_{i\in N}$ such that ${}^i x \in \mathbb{Y}$ for each player i. Let $int(\mathbb{Y})$ denote the set of completely-mixed-strategy profiles,[5] and let $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$. In this paper, we use $\mathfrak{X}$ to allude to a "structure of beliefs", to which the trembling way of "beliefs" or "conjectures" sequence for players confines. Let $\mathbf{x}^t \overset{\mathfrak{X}}{\to} \mathbf{x}$ denote a sequence $\{\mathbf{x}^t\}_{t=0}^{\infty}$ in $\mathfrak{X}$ which converges to $\mathbf{x}$ in $\mathbb{Y}^n$. (Note: We allow two players i and j to have distinct trembling sequences ${}^i x^t \rightsquigarrow {}^i x$ and ${}^j x^t \rightsquigarrow {}^j x$, respectively. We use $y^t \rightsquigarrow y$ to denote a trembling sequence $\{y^t\}_{t=0}^{\infty}$ in $int(\mathbb{Y})$ which converges to y in $\mathbb{Y}$ as $t \to \infty$.) An $\mathfrak{X}$-*assessment* is a profile-and-distributions vector $(\mathbf{x}, \boldsymbol{\mu}) \equiv ({}^i x, \mu_i)_{i\in N}$ such that there exist a sequence $\mathbf{x}^t \overset{\mathfrak{X}}{\to} \mathbf{x}$ and, for each player i, $\mu_i^t \to \mu_i$ where μ_i^t is a collection of distributions over i's information sets, derived from ${}^i x^t$ in $int(\mathbb{Y})$ using Bayes' rule. Let $B_i({}^i x, u_i)$ denote the set of player i's "locally" best responses to ${}^i x \in \mathbb{Y}$, i.e.,

$$B_i({}^i x, u_i) \equiv \left\{ y_i \in \mathbb{Y}_i : \forall h \in H_i,\ v_i\left(\left(y_h,\ {}^i x_{-h}\right), u_i\right) \geq v_i\left(\left(a_h,\ {}^i x_{-h}\right), u_i\right)\ \forall a_h \in A_h \right\}.$$

Definition 1. Let $Y \subseteq \mathbb{Y}$ and $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$.

(a) [**Perfect Rationality**] A strategy profile $y \in \mathbb{Y}$ is *perfectly rational with respect to* $(Y,\ \mathfrak{X})$ if there exists $\mathbf{x}^t \overset{\mathfrak{X}}{\to} \mathbf{x}$ such that, for each player i, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i \in B_i({}^i x^t, u_i)\ \forall t$.

[4] A set $X \subseteq \mathbb{R}^n$ is *semi-algebraic* if it is the finite union of sets of the form $\{x \in \mathbb{R}^n : f_1(x) = 0, \cdots, f_k(x) = 0$ and $g_1(x) > 0, \cdots, g_m(x) > 0\}$, where the f_i and g_j are polynomials with real coefficients. A correspondence is *semi-algebraic* if and only if its graph is a semi-algebraic set.

[5] A completely-mixed-strategy profile $y \in \mathbb{Y}$ assigns strictly positive probability to every action at every information set.

(b) **[Sequential Rationality]** A strategy profile $y \in \mathbb{Y}$ is *sequentially rational with respect to* $(Y,\ \mathfrak{X})$ if there exists an $\mathfrak{X}$-assessment $(\mathbf{x}, \boldsymbol{\mu})$ such that for all i and $h \in H_i$, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i \in \arg\max_{y_i' \in \mathbb{Y}_i} v_i\left(y_i', \left({}^i x, \mu_i\right), u_i | h\right)$.[6]

That is, a strategy profile y is perfectly rational with respect to $(Y,\ \mathfrak{X})$ if there exists a sequence $\{\mathbf{x}^t\}_{t=0}^{\infty}$ of trembling-beliefs profiles for all players in the belief structure $\mathfrak{X}$ (which converges to $\mathbf{x} = ({}^i x)_{i \in N}$ in $\mathbb{Y}^n$) such that, for each player i, the limit opponent-strategy profile ${}^i x_{-i}$ resides in the scope Y_{-i} of opponents' plausible choices and the limit strategy ${}^i x_i$ is consistent with y_i which is a "locally" best response along the trembling beliefs sequence $\{{}^i x^t\}_{t=0}^{\infty}$. Similarly, a strategy profile y is sequentially rational with respect to $(Y,\ \mathfrak{X})$ if there exists an $\mathfrak{X}$-assessment $(\mathbf{x}, \boldsymbol{\mu})$ such that, for each player i, the limit opponent-strategy profile ${}^i x_{-i}$ resides in the scope Y_{-i} and the limit strategy ${}^i x_i$ is consistent with y_i which is a "sequentially" best response at every information set $h \in H_i$. Let $PB^{\mathfrak{X}}(Y, u)$ denote the set of perfectly-rational strategy profiles with respect to $(Y,\ \mathfrak{X})$, and let $SB^{\mathfrak{X}}(Y, u)$ denote the set of sequentially-rational strategy profiles with respect to $(Y,\ \mathfrak{X})$.

We next provide two characterizations of perfect rationality and sequential rationality under a wide range of behavioral assumptions. For an extensive form Γ define

$$\mathcal{R}^{\mathfrak{X}} \equiv \left\{(\mathbf{x}, u, y) \in \mathfrak{X} \times U \times \mathbb{Y} : y_i \in B_i\left({}^i x, u_i\right)\ \ \forall i \in N\right\}.$$

That is, $(\mathbf{x}, u, y) \in R^{\mathfrak{X}}$ represents a "state" where every player is perfectly rational for payoffs $u \in U$ and "belief" $x \in X$. Since Γ is finite, $B_i\left({}^i x, u_i\right)$ is characterized by finitely many polynomial inequalities and thus semi-algebraic. By Tarski-Seidenberg Theorem, $\mathcal{R}^{\mathfrak{X}}$ is a semi-algebraic set whenever $\mathfrak{X}$ is semi-algebraic. Let $cl(\mathcal{R}^{\mathfrak{X}})$ and $vcl_U(\mathcal{R}^{\mathfrak{X}})$ denote the *closure of* $\mathcal{R}^{\mathfrak{X}}$ and *vertical closure of* $\mathcal{R}^{\mathfrak{X}}$ *(on* U*)*, respectively, i.e.,

$$\begin{aligned} cl(\mathcal{R}^{\mathfrak{X}}) &\equiv \left\{(\mathbf{x}, u, y) : \left(\mathbf{x}^t, u^t, y^t\right) \to (\mathbf{x}, u, y) \text{ and } \left(\mathbf{x}^t, u^t, y^t\right) \in \mathcal{R}^{\mathfrak{X}} \text{ for all } t\right\}; \\ vcl_U(\mathcal{R}^{\mathfrak{X}}) &\equiv \left\{(\mathbf{x}, u, y) : \left(\mathbf{x}^t, u, y^t\right) \to (\mathbf{x}, u, y) \text{ and } \left(\mathbf{x}^t, u, y^t\right) \in \mathcal{R}^{\mathfrak{X}} \text{ for all } t\right\}. \end{aligned}$$

Call "$\mathbf{x}$ is consistent with (Y, y)" if "for every player i, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$". The following proposition states that $PB^{\mathfrak{X}}(Y, u)$ and $SB^{\mathfrak{X}}(Y, u)$ are related to the closure and vertical closure of $\mathcal{R}^{\mathfrak{X}}$ (under the "consistency" requirement), respectively.

Proposition 1. *For any* $Y \subseteq \mathbb{Y}$ *and* $\mathfrak{X} \subseteq [int\,(\mathbb{Y})]^n$*, (a)* $y \in PB^{\mathfrak{X}}(Y, u) \Leftrightarrow \exists\, (\mathbf{x}, u, y) \in vcl_U(\mathcal{R}^{\mathfrak{X}})$ *s.t.* $\mathbf{x}$ *is consistent with* (Y, y)*; (b)* $y \in SB^{\mathfrak{X}}(Y, u) \Leftrightarrow \exists\, (\mathbf{x}, u, y) \in cl(\mathcal{R}^{\mathfrak{X}})$ *s.t.* $\mathbf{x}$ *is consistent with* (Y, y).

To relate to Selten's (1975) perfectness, Kreps and Wilson (1982, Proposition 6) provided a useful characterization of sequential equilibrium in terms of "payoff perturbations";

[6] Player i's expected payoff conditional on h is denoted by $v_i(y_i', \left({}^i x, \mu_i\right), u_i | h) = \Sigma_{z \in Z} u_i(z) \Pr\{z | \left(y_i',\ {}^i x_{-i}\right), \mu_i\}$, where $\Pr\{z | \left(y_i',\ {}^i x_{-i}\right), \mu_i\}$ is the probability that z is reached conditionally on h under $\left(y_i',\ {}^i x_{-i}\right)$ and μ_i.

they relaxed Selten's criterion by allowing some (vanishingly) small uncertainty on the part of players' payoffs. BZ94 offered an alternative characterization of sequential equilibrium in terms of "perturbed games". Proposition 1 provides two fundamental characterizations of perfect rationality and sequential rationality under a broader range of behavioral assumptions; for example, if Y is restricted to a singleton set and $\mathfrak{X}$ is restricted to a "common" trembling-beliefs sequence in $\{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^i x = {}^j x \text{ for all } i \neq j\}$, Proposition 1(b) yields an analogy of Kreps and Wilson's (1982) characterization of sequential equilibrium.

4 Generic equivalence theorem

In this section, we establish a general "generic" equivalence between perfect rationality and sequential rationality. Our proof is based on the fundamental structure of semi-algebraic set: each semi-algebraic set has only a finite number of open connected components, and has a well-defined dimension. The following property of semi-algebraic sets is crucial in our paper.

Generic Local Triviality [Hardt (1980); Bochnak, Coste and Roy (1987, Corollary 9.3.2)]. *Let B and U be semi-algebraic sets and let $f : B \to U$ be a continuous, semi-algebraic function. There is a (relatively) closed, lower-dimensional semi-algebraic ("critical") subset $U^0 \subset U$ such that for each of the finite number of (relatively) open connected components U^k of $U \backslash U^0$ there is a semi-algebraic ("fiber") set C^k and a semi-algebraic homeomorphism $\varphi^k : U^k \times C^k \to f^{-1}\left(U^k\right)$ such that $f\left(\varphi^k(u,c)\right) = u$ for all $u \in U^k$ and $c \in C^k$.*

Generic Local Triviality implies that, for any semi-algebraic set, the closure and vertical closure are almost the same: that is, the difference between the closure and vertical closure of a semi-algebraic set is lower-dimensional. Formally,

Proposition 2. *Let $X \subseteq \mathbb{R}^{n+m}$ be a semi-algebraic set. (a) $cl(X)$ and $vcl_{\mathbb{R}^n}(X)$ are semi-algebraic. (b) There exists a lower-dimensional semi-algebraic subset $X^0_{\mathbb{R}^n} \subset \mathbb{R}^n$ such that $cl(X) \backslash vcl_{\mathbb{R}^n}(X) \subseteq X^0_{\mathbb{R}^n} \times \mathbb{R}^m$.*

By Propositions 1 and 2, we obtain the central result of this paper: a general "generic" equivalence theorem between perfect rationality and sequential rationality.

Theorem 1. *Consider an extensive form Γ. For any semi-algebraic set $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$, there is a (relatively) closed, lower-dimensional semi-algebraic subset $U^0 \subset U$ such that, for all $u \in U \backslash U^0$, $PB^{\mathfrak{X}}(Y,u) = SB^{\mathfrak{X}}(Y,u)\ \forall Y \subseteq \mathbb{Y}$. Furthermore, if $\mathfrak{X} = [int(\mathbb{Y})]^n$, there is a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that, for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$, $PB^{\mathfrak{X}}(Y,u) = SB^{\mathfrak{X}}(Y,u)\ \forall Y \subseteq \mathbb{Y}$.*

Theorem 1 establishes a fundamental and elementary "generic" equivalence between perfect rationality and sequential rationality. More specifically, the equivalence holds for all payoff vectors outside a lower-dimensional subset $U^0 \subset U = \mathbb{R}^{N\times Z}$; under a belief structure in product form: $\mathfrak{X} = [int(\mathbb{Y})]^n$, the equivalence holds for all assigned payoffs for each player outside a lower-dimensional subset $V^0 \subset U_i = \mathbb{R}^Z$, rather than a lower-dimensional subset $U^0 \subset U$. In this paper, we consider two kinds of belief structures used in extensive games:

1. $\mathfrak{X} = [int(\mathbb{Y})]^n$. Under this belief structure, different players i and j are allowed to have distinct trembling sequences ${}^ix^t \rightsquigarrow {}^ix$ and ${}^jx^t \rightsquigarrow {}^jx$. For all $Y \subseteq \mathbb{Y}$, we denote $PB(Y,u) \equiv PB^{\mathfrak{X}}(Y,u)$ and $SB(Y,u) \equiv SB^{\mathfrak{X}}(Y,u)$.

2. $\mathfrak{X} \equiv \{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^ix = {}^jx \text{ for all } i \neq j\}$. Under this belief structure, different players i and j are required to have a common trembling sequence $x^t \rightsquigarrow x$. For all $Y \subseteq \mathbb{Y}$, we denote $PB^*(Y,u) \equiv PB^{\mathfrak{X}}(Y,u)$ and $SB^*(Y,u) \equiv SB^{\mathfrak{X}}(Y,u)$; in particular, we write $PB^*(y,u)$ and $SB^*(y,u)$ respectively for $PB^*(\{y\},u)$ and $SB^*(\{y\},u)$, for simplicity.

Definition 2. In game $\Gamma(u)$, we define

(a) [**Perfect Equilibrium**] A strategy profile y is a *perfect equilibrium* if $y \in PB^*(y,u)$, i.e., there exists a (common) sequence $y^t \rightsquigarrow y$ such that for all $h \in H$, $y_h \in \arg\max_{y'_h \in \mathbb{Y}_h} v_i\left(\left(y'_h, y^t_{-h}\right), u_i\right) \; \forall t$.

(b) [**Sequential Equilibrium**] A strategy profile y is a *sequential equilibrium* if $y \in SB^*(y,u)$, i.e., there exists a (common) assessment (y,μ) such that for all $i \in N$ and $h \in H_i$, $y_i \in \arg\max_{y'_i \in \mathbb{Y}_i} v_i(y'_i, (y,\mu), u_i|h)$.

(c) [**Perfect Rationalizability**] A strategy profile y is *perfectly rationalizable* if it is supported by a perfectly rationalizable set $Y \subseteq \mathbb{Y}$, i.e., $y \in Y \subseteq PB\,(Y,u)$.

(d) [**Sequential Rationalizability**] A strategy profile y is *sequentially rationalizable* if it is supported by a sequentially rationalizable set $Y \subseteq \mathbb{Y}$, i.e., $y \in Y \subseteq SB\,(Y,u)$.

Definition 2(a) is Selten's (1975) notion of perfect equilibrium. Definition 2(b) is Kreps and Wilson's (1982) notion of sequential equilibrium. Definition 2(c) is a variant of Greenberg et al.'s (2009) notion of null MACA, which, if allows for correlations, is equivalent to Herings and Vannetelbosch's (1999) definition of "weakly perfect rationalizability" in simultaneous-move games. Definition 2(d) is a variant of Dekel et al.'s (1999, 2002) sequential rationalizability (with point beliefs).

Remark 1. For simplicity, we consider only point beliefs over opponents' strategies in the notion of rationalizability (see Bernheim (1984)). Apparently, since every singleton of a perfectly/sequentially rationalizable strategy profile is a "weak" version of perfect/sequential equilibrium (by allowing distinct trembling sequences for different players),[7] every perfect/sequential equilibrium must be perfectly/sequentially rationalizable.

For game $\Gamma(u)$, we need to introduce the following notation:

$PE(u)$: set of perfect equilibria
$SE(u)$: set of sequential equilibria
$WPE(u)$: set of "weakly" perfect equilibria
$WSE(u)$: set of "weakly" sequential equilibria
$PR(u)$: set of perfectly rationalizable strategy profiles
$SR(u)$: set of sequentially rationalizable strategy profiles

According to Theorem 1, $PB^{\mathfrak{X}}(Y, u)$ and $SB^{\mathfrak{X}}(Y, u)$ generically coincide for any arbitrary $Y \subseteq Y$. Note that the perfect and sequential notions of equilibrium and rationalizability are based on the basic assumptions of perfect rationality and sequential rationality, we obtain "generic" equivalence results for equilibrium and rationalizability, as immediate corollaries of Theorem 1.

Corollary 1. *Consider an extensive form* Γ*. There is a (relatively) closed, lower-dimensional semi-algebraic subset* $V^0 \subset U_i = \mathbb{R}^Z$ *such that, for all* $u \in \Pi_{i\in N}(U_i \backslash V^0)$*,* $Y(u)$ *is a sequentially rationalizable set in* $\Gamma(u)$ *iff* $Y(u)$ *is a perfectly rationalizable set in* $\Gamma(u)$*. Moreover, a sequentially-rationalizable-set correspondence* $Y(\cdot)$ *(i.e.,* $Y : U \rightrightarrows \mathbb{Y}$ *such that* $Y(u) \subseteq SB(Y(u), u)\ \forall u \in U$*) is perfectly rationalizable for all* $u \in U$ *at which* $SB(Y(\cdot), \cdot)$ *is lower hemi-continuous and* $PB(Y(\cdot), \cdot)$ *is upper hemi-continuous.*

In particular, $PR(u) = SR(u)$ *and* $WPE(u) = WSE(u)$ *for all* $u \in \Pi_{i\in N}(U_i \backslash V^0)$*. Moreover,* $PR(u) = SR(u)$ *for all* $u \in U$ *at which correspondence* $SR(\cdot)$ *is lower hemi-continuous and* $PR(\cdot)$ *is upper hemi-continuous;* $WPE(u) = WSE(u)$ *for all* $u \in U$ *at which correspondence* $WSE(\cdot)$ *is lower hemi-continuous and* $WPE(\cdot)$ *is upper hemi-continuous.*

Remark 2. The class of symmetric games or zero-sum two-person games has the same dimension of U_i, because payoff vectors are fully determined by a particular player i's payoffs. Consequently, Corollary 1 implies that, in the class of symmetric games or zero-sum two-person games, the equivalence holds for all "generically" assigned payoffs $u_i \in U_i \backslash V^0$ (for the particular player i).

A critial assumption of the beliefs structure in Corollary 1 is: $\mathfrak{X} = [int(\mathbb{Y})]^n$; accordingly, we allow two players i and j to have distinct trembling-beliefs sequences ${}^i x^t \rightsquigarrow {}^i x$ and

[7] That is, different players may not necessarily have the same beliefs on how players "tremble". Fudenberg and Tirole (1991, p.341) pointed out, "Why should all players have the same theory to explain deviations that, after all, are either probability-0 events or very unlikely, depending on one's methodological point of view? The standard defense is that this requirement is in the spirit of equilibrium analysis, since equilibrium supposes that all players have common beliefs about the others' strategies. Although this restriction is usually imposed, we are not sure that we find it convincing."

${}^j x^t \rightsquigarrow {}^j x$, respectively. If we impose a stronger assumption of the beliefs structure, i.e., $\mathfrak{X}$ is restricted to a "common" trembling-beliefs sequence for all players in $\{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^i x = {}^j x \text{ for all } i \neq j\}$, Theorem 1 yields BZ94's (Theorem 4) "generic" equivalence result for perfect and sequential equilibria.

Corollary 2. *Consider an extensive form Γ. There is a (relatively) closed, lower-dimensional semi-algebraic subset $U^0 \subset U$ such that $PE(u) = SE(u)$ for all $u \in U \backslash U^0$. Moreover, $PE(u) = SE(u)$ for all $u \in U$ at which correspondence $SE(\cdot)$ is lower hemi-continuous and $PE(\cdot)$ is upper hemi-continuous.*

Normal forms are a special case of extensive forms with simultaneous moves. Corollary 3 asserts that, in any normal form, iterated elimination of weakly dominated strategies (IEWDS) is generically an order-independent procedure which is equivalent to iterated elimination of strictly dominated strategies (IESDS).[8]

Corollary 3. *Consider a normal form Γ. There exists a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$, every IEWDS procedure is an IESDS procedure; hence IEWDS is (generically) an order-independent procedure.*

In the context of extensive games, Greenberg et al. (2009) presented a unified solution concept of "mutually acceptable course of action (MACA)" for situations where "perfectly" rational individuals with different beliefs agree to a shared course of action. We end this section by establishing a "generic" equivalence between perfect-MACA and sequential-MACA, as an immediate corollary of Theorem 1. In doing so, we extend the simple version of point beliefs to a more complicated version of (uncorrelated) beliefs in extensive games. Following Dekel et al. (2002), we say a strategy y_i of player i is in the "extensive-form convex hull" of $Y_i \subseteq \mathbb{Y}_i$, denoted by $co^e(Y_i)$, if there is a finite set $\{y_i^1, ..., y_i^M\} \subseteq Y_i$, with trembling sequences $(y_i^{m,t})_{m=1}^M \rightsquigarrow (y_i^m)_{m=1}^M$ and a sequence $(\alpha^{m,t})_{m=1}^M \rightarrow \alpha$ of distributions on $\{1, ..., M\}$, such that y_i^t generated by the convex combination $\Sigma_{m=1}^M \alpha^{m,t} y_i^{m,t}$ (in terms of "realization outcomes") converges to y_i. Let $co^e(Y) \equiv \Pi_{i \in N} co^e(Y_i)$. In the spirit of Greenberg et al. (2009), we introduce the "perfect" and "sequential" notions of MACA.[9]

Definition 3. In game $\Gamma(u)$, a course of action $\sigma(u) = (\sigma_h(u))_{h \in H}$, with $\sigma_h(u) \in \mathbb{Y}_h \cup \{\emptyset\}$, is a *perfect-MACA* (or *sequential-MACA*) if there is $Y = \Pi_{i \in N} Y_i \subseteq \mathbb{Y}$ supporting $\sigma(u)$, i.e.,

(i) for all $h \in H$, if $\sigma_h(u) \neq \emptyset$, then $y_h = \sigma_h(u)$ for all $y \in Y$;

[8] Rochet (1980) showed that, in finite games with perfect information, the unique backward induction payoff is the same as the unique payoff from iterated weak dominance; see also Marx and Swinkels (1996) for extensive discussions.

[9] The formulation of an "extensive-form convex hull" purports to deal with the notorious problem of imperfection under subjective uncertainty over (behavior) strategies; cf. Dekel et al. (2002) and Greenberg et al. (2009) for more discussions. For the purpose of this paper, we here adopt Dekel et al.'s (2002) definition of "extensive-form convex hull" to define the notions of perfect-MACA and sequential-MACA (within Greenberg et al.'s (2009) framework of MACA).

(ii) $Y \subseteq PB(co^e(Y), u)$ (or $Y \subseteq SB(co^e(Y), u)$).

The following corollary asserts that the notions of perfect-MACA and sequential-MACA are generically equivalent.

Corollary 4. *Consider an extensive form* Γ. *There is a (relatively) closed, lower-dimensional semi-algebraic subset* $V^0 \subset U_i = \mathbb{R}^Z$ *such that for all* $u \in \Pi_{i \in N}(U_i \backslash V^0)$, *the set of perfect-MACAs coincides with the set of sequential-MACAs and, moreover, a sequential-MACA* $\sigma(u)$ *is supported by* Y *iff* $\sigma(u)$ *is a perfect-MACA supported by* Y.

Remark 3. Greenberg et al. (2009) demonstrated that, by varying the degree of completeness of the underlying course of action, MACA can be related to many commonly used game-theoretic solutions, such as equilibrium, self-confirming equilibrium, and rationalizability. More specifically,

(i) If $\sigma(u)$ is a "complete" MACA (which satisfies $\sigma_h(u) \neq \varnothing \ \forall h \in H$), Corollary 4 yields the "generic" equivalence result between "weakly" sequential equilibria and "weakly" perfect equilibria in Corollary 1 (cf. Greenberg et al.'s (2009) Claim 3.1.1),

(ii) If $\sigma(u)$ is a "null" MACA (which satisfies $\sigma_h(u) = \varnothing \ \forall h \in H$), Corollary 4 yields a "generic" equivalence result between Dekel et al.'s (1999, 2002) sequential rationalizability and perfect rationalizability with trembling beliefs in an "extensive-form convex hull", rather than the simple version of point beliefs used in Corollary 1 (cf. Greenberg et al.'s (2009) Claim 3.3.1), and

(iii) If $\sigma(u)$ is a "path" MACA (which satisfies $\sigma_h(u) \neq \varnothing$ whenever h is reached with positive probability under $\sigma(u)$), Corollary 4 yields a "generic" equivalence result between Dekel et al.'s (1999, 2002) sequentially rationalizable self-confirming equilibrium (SRSCE) and Greenberg et al.'s (2009) path MACA.

We would also like to point out that Dekel et al. (1999, Footnote 4) expected this kind of "generic" equivalence, but they offered no formal analysis of this issue. We thereby offer such a formal analysis from this perspective.

5 Concluding remarks

Blume and Zame (1994) strengthened Kreps and Wilson's (1982) result and showed that, for almost all assignments of payoffs to the terminal nodes, the sets of sequential and perfect equilibria are identical. In this paper, we have extended BZ94's result to more general settings of strategic interactions. We have formulated and proved a general "generic" equivalence theorem between perfect rationality and sequential rationality in extensive games. More specifically, we have presented a general framework to accommodate many structures of beliefs discussed in the literature and shown that the difference between perfectly and sequentially rational correspondences occurs only in a lower-dimensional payoffs set. We have also demonstrated that we can obtain a variety of generic equivalence results for various kinds of solution concepts such as equilibrium, rationalizability, iterated dominance and MACA, as corollaries of our general "generic" equivalence theorem (Theorem 1). The study of this paper helps deepen our understanding of the relationship between perfectly and sequentially rational strategic behavior with different structures of beliefs.

In this paper, we have followed Dekel et al. (1999, 2002) and Greenberg et al. (2009) to adopt a simple and convenient way of defining perfect/sequential equilibrium and rationalizability by using "trembling conjectures" and present a unified framework for the study of the "generic" relationship between perfectly and sequentially rational strategic behavior. Alternatively, one may follow BZ94's approach to analyze perfectly and sequentially rational strategic behavior by using "perturbed games". However, there is no formal formulation of perfect/sequential rationalizability for extensive games, in terms of "perturbed games", although Bernheim (1984, pp.1021-1022) outlined such a notion of perfect rationalizability in normal form games. Herings and Vannetelbosch (1999, Example G7) showed that, unlike the notion of perfect equilibrium, there are different definitions of perfect rationalizability by using "trembling conjectures" or "perturbed games" (cf. also Börgers (1994)). In particular, the alternative definition of perfect/sequential rationalizability by using "perturbed games" may suffer Fudenberg and Tirole's (1991) criticism: it implicitly requires that all players have the same theory to form common "trembling conjectures", as illustrated by the following example.

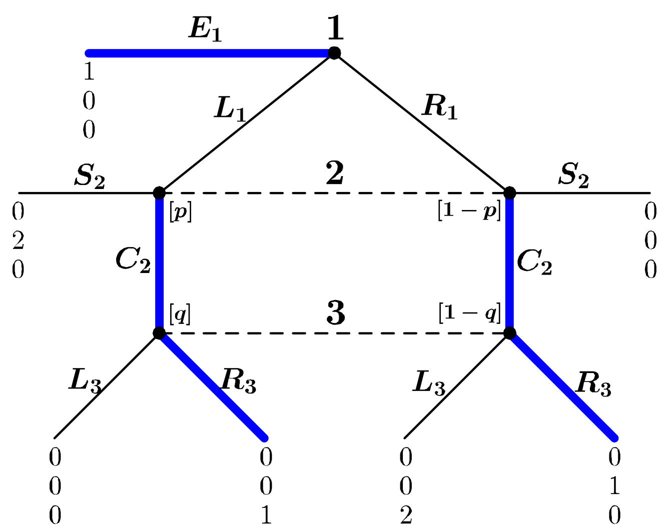

Fig. 2. A three-person game.

It is easy to see that the strategy profile $y = (E_1, C_2, R_3)$, marked by bold lines in Figure 2, is a "weakly" sequential/perfect equilibrium and, by Corollary 1, y is sequentially/perfectly rationalizable for almost all assignments of payoffs to the terminal nodes. But, the profile y

is not sequentially/perfectly rationalizable in terms of "perturbed games". To see this, note that (i) in any perturbed game, because E_1 strictly dominates L_1 and R_1, there is a unique rationalizable strategy for player 1 – i.e., playing L_1 and R_1 with the minimum probabilities specified in the perturbed game, and (ii) player 2 and 3 must hold common "trembling conjectures" in commonly known "perturbed games". But, C_2 is sequentially rational only if $p \leq 1/3$; R_3 is sequentially rational only if $q \geq 2/3$. Subsequently, the profile (C_2, R_3) cannot be sequentially/perfectly rationalizable in terms of "perturbed games". This argument is valid for a neighborhood of the payoffs to the terminal nodes.[10] Since this kind of implicit requirement of common "trembling conjectures" appears to be less convincing and arguable especially in a non-equilibrium setting, we do not use this alternative way of formulating perfectly/sequentially rational strategic behavior in this paper.

As we have emphasized, unlike BZ94's approach, our analysis of this paper does not rely directly on semi-algebraic properties of specific solutions concepts (e.g., the semi-algebraic structure of perfect and sequential equilibrium correspondences in BZ94). Instead, our approach of this paper is based upon the primitive set $\mathcal{R}^{\mathfrak{X}}$ of "perfectly-rational" states, which is naturally semi-algebraic with different structures of beliefs, so that it is feasible and applicable to various solution concepts discussed in the literature. We believe that our general "generic" equivalence theorem provides a useful and complementary way for the study of the relationship between perfectly and sequentially rational strategic behavior in complex environments.[11]

Finally, we would like to mention that, in contrast to BZ94's approach to complete-information games through perturbations on payoffs, Weinstein and Yildiz (2007) took a different approach to generic properties of rational strategic behavior and showed, in the framework of incomplete-information games with richness assumption, a generic uniqueness result for the structure of rationalizability by perturbing (in the product topology of the universal type space) the beliefs of the type. It is intriguing to extend the analysis of this paper to a general situation by allowing perturbations both on payoffs and the beliefs of the type. We leave it for future research.

[10] This example also shows that the notions of perfect Bayesian equilibrium and sequential equilibrium are generically different, because (E_1, C_2, R_3) is a perfect Bayesian equilibrium.

[11] We note that our approach of this paper is also applicable to the alternative definitions of perfectly and sequentially rational strategic behavior by using "perturbed games". In doing so, we need to consider a more elaborated set of perfectly-rational states with "game perturbations" and then obtain an analogous "generic" equivalence.

6 Appendix: Proofs

The following lemma establishes a relationship between perfect rationality and sequential rationality: that is, sequential rationality can be characterized by perfect rationality against payoff perturbations.

Lemma 1. *For any $Y \subseteq \mathbb{Y}$ and $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$, $y \in SB^{\mathfrak{X}}(Y,u)$ iff there exist $u^t \to u$ and $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ such that $\mathbf{x}$ is consistent with (Y,y) and for each player i, $y_i \in B_i({}^ix^t, u_i^t)$ $\forall t$.*

Proof of Lemma 1. "$\Leftarrow$": Let $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ and $\mathbf{x}$ be consistent with (Y,y). Without loss of generality, assume $(\mathbf{x}^t, \boldsymbol{\mu}^t) \xrightarrow{\mathfrak{X}} (\mathbf{x}, \boldsymbol{\mu})$, where $\boldsymbol{\mu}^t$ is derived from $\mathbf{x}^t$ using Bayes' rule. Consider player i, suppose that there exist $u_i^t \to u_i$ such that $y_i \in B_i({}^ix^t, u_i^t)$ for all t. Then, for all $h \in H_i$ and all t, $v_i\left((y_h,\ {}^ix_{-h}^t), u_i^t|h\right) \geq v_i\left((y_h',\ {}^ix_{-h}^t), u_i^t|h\right)$ $\forall y_h' \in \mathbb{Y}_h$.[12] Since $v_i((y_h,\cdot),\cdot|h)$ is continuous, $v_i\left((y_h,\ {}^ix_{-h}), u_i|h\right) \geq v_i\left((y_h',\ {}^ix_{-h}), u_i|h\right)$. By the one deviation property (see, e.g., Osborne and Rubinstein (1994, p.227)), for all $h \in H_i$, $v_i\left((y_i,\ {}^ix_{-i}), u_i|h\right) \geq v_i\left((y_i',\ {}^ix_{-i}), u_i|h\right)$ $\forall y_i' \in \mathbb{Y}_i$. That is, $y_i \in SB_i({}^ix, u_i)$ for all i and thus $y \in SB^{\mathfrak{X}}(Y,u)$.

"$\Rightarrow$": Let $y \in SB^{\mathfrak{X}}(Y,u)$. Then, for each player i there is $(\mathbf{x}^t, \boldsymbol{\mu}^t) \xrightarrow{\mathfrak{X}} (\mathbf{x}, \boldsymbol{\mu})$ such that for all i, ${}^ix_{-i} \in Y_{-i}$ and ${}^ix_i = y_i$ is sequentially optimal to assessment $({}^ix, \mu_i)$. Clearly, ${}^ix^t \to {}^ix$. We proceed to construct a payoff sequence $u_i^t \to u_i$ such that $y_i \in B_i({}^ix^t, u_i^t)$ for all t.

Since Γ is finite and perfect recall, we can define a (finite) partition $\left\{H_i^l\right\}_{l=1}^L$ of a set H_i as follows: $H_i^l \equiv \left\{h \in H_i \backslash \cup_{\ell<l} H_i^\ell : \text{no } h' \in \left[H_i \backslash \cup_{\ell<l} H_i^\ell\right] \backslash h \text{ is reached by } h\right\}$ for all $l \geq 1$. Let $u_i^{t,0} \equiv u_i$. For all t and $l = 1, \cdots, L$, $u_i^{t,l}$ is define recursively as follows:

$$u_i^{t,l}(z) \equiv \begin{cases} u_i^{t,l-1}(z) + \delta_{a_h^*}^t, & \text{if } z \text{ is not precluded by } a_h^* \in \text{support}(y_h) \text{ from } h \\ u_i^{t,l-1}(z), & \text{otherwise} \end{cases},$$

where $h \in H_i^l$, $\text{support}(y_h) = \{a_h \in A_h : y_h(a_h) > 0\}$ and

$$\delta_{a_h^*}^t = \max_{a_h \in A_h} v_i\left(\left(a_h,\ {}^ix_{-h}^t\right), u_i^{t,l-1}|h\right) - v_i\left(\left(a_h^*,\ {}^ix_{-h}^t\right), u_i^{t,l-1}|h\right).$$

Therefore, for $l = 1, \cdots, L$, $y_h \in B_h\left({}^ix_{-h}^t, u_i^{t,l}\right)$ $\forall h \in H_i^l$.[13] Since Γ is perfect recall, for all $a_h, a_h' \in A_h$, $v_i\left(\left(a_h,\ {}^ix_{-h}^t\right), u_i^{t,l+1}|h\right) - v_i\left(\left(a_h',\ {}^ix_{-h}^t\right), u_i^{t,l+1}|h\right) = v_i\left(\left(a_h,\ {}^ix_{-h}^t\right), u_i^{t,l}|h\right) - v_i\left(\left(a_h',\ {}^ix_{-h}^t\right), u_i^{t,l}|h\right)$. By induction on l, we have $y_h \in B_h\left({}^ix_{-h}^t, u_i^{t,l}\right)$ $\forall h \in \cup_{\ell=1}^l H_i^\ell$. Hence, $y_h \in B_h\left({}^ix_{-h}^t, u_i^{t,L}\right)$ $\forall h \in H_i$, i.e., $y_i \in B_i({}^ix^t, u_i^{t,L})$.

It remains to show $u_i^{t,L} \to u_i^L = u_i$. We prove this by induction on l. Clearly, $u_i^{t,0} \to u_i$ trivially holds. Suppose $u_i^{t,\ell} \to u_i^\ell = u_i$ for $\ell \leq l-1$. By construction of $u_i^{t,l}$, it suffices to show $\delta_{a_h^*}^t \to 0$ $\forall h \in H_i^l$. Let $\hat{a}_h \in \arg\max_{a_h \in A_h} v_i\left(\left(a_h,\ {}^ix_{-h}^t\right), u_i^{t,l-1}|h\right)$. Because of the

[12] For any $y_i' \in \mathbb{Y}_i$, we define $v_i\left(\left(y_i',\ {}^ix_{-i}^t\right), u_i^t|h\right) \equiv v_i\left(y_i', \left({}^ix^t, \mu_i^t\right), u_i^t|h\right)$ and $v_i\left(\left(y_i',\ {}^ix_{-i}\right), u_i|h\right) \equiv v_i\left(y_i', \left({}^ix, \mu_i\right), u_i|h\right)$.

[13] For $y_{-h} \in \mathbb{Y}_{-h}$ and $u_i \in U_i$, define $B_h(y_{-h}, u_i) \equiv \{y_h \in \mathbb{Y}_h : v_i((y_h, y_{-h}), u_i) \geq v_i((a_h, y_{-h}), u_i)\ \forall a_h \in A(h)\}$.

continuity of v_i, for any $\varepsilon > 0$ there is a sufficiently large T such that, for all $t > T$,

$$\begin{aligned} v_i\left(\left(\hat{a}_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) - v_i\left(\left(\hat{a}_h,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) &< \varepsilon; \\ v_i\left(\left(a^*_h,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) - v_i\left(\left(a^*_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) &< \varepsilon. \end{aligned}$$

Since $y_h \in B_h\left({}^i x_{-h}, u_i\right)$ and, by induction assumption, $u_i = u_i^{l-1}$,

$$v_i\left(\left(\hat{a}_h,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) - v_i\left(\left(a^*_h,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) \leq 0.$$

Therefore, $v_i\left(\left(\hat{a}_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) - v_i\left(\left(a^*_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) < 2\varepsilon$, i.e., $\delta^t_{a^*_h} \to 0$. □

Proof of Proposition 1. Suppose $y \in PB^{\mathfrak{X}}(Y, u)$. Then, there is $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ such that $(\mathbf{x}^t, u, y) \in \mathcal{R}^{\mathfrak{X}}$ for all t; ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$ for all i. Clearly, $(\mathbf{x}^t, u, y) \to (\mathbf{x}, u, y)$. Thus $(\mathbf{x}, u, y) \in vcl_{U_i}\left(\mathcal{R}^{\mathfrak{X}}\right)$. Conversely, suppose $(\mathbf{x}, u, y) \in vcl_{U_i}\left(\mathcal{R}^{\mathfrak{X}}\right)$, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$ for all i. Then there exists a sequence $(\mathbf{x}^t, u, y^t) \in \mathcal{R}^{\mathfrak{X}}$ converging to $(\mathbf{x}, u, y)$. Since Γ is finite and $y^t \to y$, there is a sufficiently large T such that, for all $t \geq T$, $a^*_h \in$support(y_h) implies $a^*_h \in$support(y^t_h) and $a^*_h \in B_h\left({}^i x^t, u_i\right)$ $\forall h \in H_i$. Therefore, $y_i \in B_i\left({}^i x^t, u_i\right)$ for all i and $t \geq T$. That is, $y \in PB^{\mathfrak{X}}(Y, u)$. Since a similar argument holds true with payoff perturbations, by using Lemma 1, Proposition 1(b) is valid. □

Proof of Proposition 2. (a) $vcl_{\mathbb{R}^n}(X)$ can be rewritten as

$$\left\{(a,b) \in \mathbb{R}^n \times \mathbb{R}^m : \forall \varepsilon > 0,\ \exists\,(a,b') \in X \text{ s.t. } \|b - b'\| < \varepsilon\right\}.$$

Since X is semi-algebraic, it follows from Tarski-Seidenberg Theorem that $vcl_{\mathbb{R}^n}(X)$ is also semi-algebraic. Similarly, $cl\,(X)$ is semi-algebraic.

(b) Denote $\pi : X \to X_{\mathbb{R}^n}$ as the projection function of X onto the first n coordinates, where $X_{\mathbb{R}^n} = \{a \in \mathbb{R}^n : \exists b \in \mathbb{R}^m \text{ s.t. } (a,b) \in X\}$. X is endowed with the relative topology of the usual product topology $\mathbb{R}^n \times \mathbb{R}^m$. Then π is continuous and its graph is $\{((a,b), a) : (a,b) \in X\}$, which is semi-algebraic. Thus we can apply Generic Local Triviality to π. Denote A^0 as the critical set with $\dim A^0 < n$, $X_{\mathbb{R}^n} \backslash A^0 = \cup_k A^k$ as the decomposition into finitely many connected open components, C^k as the fiber for A^k. Denote φ^k as the semi-algebraic homeomorphism between $A^k \times C^k$ and $\pi^{-1}\left(A^k\right)$. Moreover, all φ^k satisfy the following condition:

$$\forall a \in A^k,\ \forall c \in C^k,\ \pi\left(\varphi^k(a,c)\right) = a. \tag{\#}$$

Let $X^0_{\mathbb{R}^n} = \cup_k \left(cl\left(A^k\right)\backslash A^k\right) \cup cl\left(A^0\right)$. Suppose $(a,b) \in cl\,(X)$ and $a \in \mathbb{R}^n \backslash X^0_{\mathbb{R}^n}$. By the decomposition of $X_{\mathbb{R}^n}$, $X = \cup_k \pi^{-1}\left(A^k\right) \cup \pi^{-1}\left(A^0\right)$, then there is a sequence (a^t, b^t) in $\pi^{-1}\left(A^k\right)$ for some component A^k and $(a^t, b^t) \to (a,b)$. Denote $\left(\varphi^k\right)^{-1}(a^t, b^t) = (\hat{a}^t, c^t) \in A^k \times C^k$, then $\pi\left(\varphi^k(\hat{a}^t, c^t)\right) = \pi(a^t, b^t) = a^t$. By (#), $a^t = \hat{a}^t$. Since $a \in \mathbb{R}^n \backslash X^0_{\mathbb{R}^n}$, $a \notin cl\left(A^k\right)\backslash A^k$. Since $a^t \to a$, $a \in cl\left(A^k\right)$. Thus $a \in A^k$. Then (a, c^t) is a sequence in $A^k \times C^k$. Denote $\varphi^k(a, c^t) = \left(a, \hat{b}^t\right)$ for all t. Since $\|(a,c^t) - (a^t, c^t)\| \to 0$ and φ^k

is continuous, $\left\|\left(a,\hat{b}^t\right)-\left(a^t,b^t\right)\right\| = \left\|\varphi^k\left(a,c^t\right)-\varphi^k\left(a^t,c^t\right)\right\| \to 0$. Therefore, $\left(a,\hat{b}^t\right) \to (a,b)$, i.e., $(a,b) \in vcl_{\mathbb{R}^n}\left(\pi^{-1}\left(A^k\right)\right) \subseteq vcl_{\mathbb{R}^n}(X)$. Since $\dim\left[cl\left(A^0\right)\right] = \dim A^0 < n$ and $\dim\left[cl\left(A^k\right)\backslash A^k\right] < n$, $\dim X^0_{\mathbb{R}^n} < n$. □

Proof of Theorem 1. Since $\mathfrak{X}$ is semi-algebraic, $\mathcal{R}^{\mathfrak{X}}$ is a semi-algebraic set by Tarski-Seidenberg Theorem. Applying Proposition 2 to $\mathcal{R}^{\mathfrak{X}}$, there exists a closed semi-algebraic subset $U^0 \subset U$ with $\dim U^0 < \dim U$ such that $cl(\mathcal{R}^{\mathfrak{X}})\backslash vcl_U(\mathcal{R}^{\mathfrak{X}}) \subseteq \mathbb{Y}^n \times U^0 \times \mathbb{Y}$. Therefore, for all $u \in U\backslash U^0$,

$$(\mathbf{x},u,y) \in cl(\mathcal{R}^{\mathfrak{X}}) \Leftrightarrow (\mathbf{x},u,y) \in vcl_U(\mathcal{R}^{\mathfrak{X}}). \qquad (*)$$

Consider an arbitrary set $Y \subseteq \mathbb{Y}$. For all $u \in U\backslash U^0$, we have

$$\begin{aligned} y \in SB^{\mathfrak{X}}(Y,u) &\overset{\text{Proposition 1}}{\Longleftrightarrow} \exists\,(\mathbf{x},u,y) \in cl(\mathcal{R}^{\mathfrak{X}}) \text{ s.t. } {}^ix_{-i} \in Y_{-i} \text{ and } {}^ix_i = y_i\ \forall i \\ &\overset{(*)}{\Longleftrightarrow} \exists\,(\mathbf{x},u,y) \in vcl_U(\mathcal{R}^{\mathfrak{X}}) \text{ s.t. } {}^ix_{-i} \in Y_{-i} \text{ and } {}^ix_i = y_i\ \forall i \\ &\overset{\text{Proposition 1}}{\Longleftrightarrow} y \in PB^{\mathfrak{X}}(Y,u). \end{aligned}$$

Now let $\mathfrak{X} = [int(\mathbb{Y})]^n$. Then, $\mathcal{R}^{\mathfrak{X}} = \Pi_{i\in N}\mathcal{R}^{\mathfrak{X}}_i$ where $\mathcal{R}^{\mathfrak{X}}_i \equiv \{({}^ix,u_i,y_i) \in int(\mathbb{Y}) \times U_i \times \mathbb{Y}_i : y_i \in B_i({}^ix,u_i)\}\ \forall i \in N$. Applying Proposition 2 to each set $\mathcal{R}^{\mathfrak{X}}_i$, there exists a closed semi-algebraic subset $U^0_i \subset U_i$ with $\dim U^0_i < \dim U_i$ such that $cl(\mathcal{R}^{\mathfrak{X}}_i)\backslash vcl_{U_i}(\mathcal{R}^{\mathfrak{X}}_i) \subseteq \mathbb{Y} \times U^0_i \times \mathbb{Y}_i$. Define $V^0 \equiv \cup_{i\in N}U^0_i$. Therefore, for all $u \in \Pi_{i\in N}(U_i\backslash V^0)$, the identity $(*)$ holds. The rest of Theorem 1 follows similarly. □

In order to show Corollaries 1 and 2, we need the following lemma.

Lemma 2. *Let $Y : U \rightrightarrows \mathbb{Y}$ and $Y' : U \rightrightarrows \mathbb{Y}$. Suppose that $U^0 \equiv \{u \in U : Y(u) \neq Y'(u)\}$ is a lower dimensional subset of U. Then $Y(u) \subseteq Y'(u)$ for all $u \in U$ at which $Y(\cdot)$ is lower hemi-continuous and $Y'(\cdot)$ is upper hemi-continuous.*

Proof of Lemma 2. Since U^0 is lower-dimensional, U^0 contains no open set in U. Let $u \in U$. Therefore, we can find a sequence $\{u^t\}_{t=1}^{\infty}$ in $U\backslash U^0$ such that $u^t \to u$ and $Y(u^t) = Y'(u^t)$ for all t. If $y \in Y(u)$, by lower hemi-continuity of $Y(\cdot)$, there exists a subsequence $u^{t_k} \to u$ such that $y^k \to y$ and $y^k \in Y(u^{t_k}) = Y'(u^{t_k})$. Since correspondence $Y'(\cdot)$ is upper hemi-continuous, $y \in Y'(u)$. That is, $Y(u) \subseteq Y'(u)$. □

Proof of Corollary 1. Let $Y(u)$ be a sequentially rationalizable set in $\Gamma(u)$, i.e., $Y(u) \subseteq SB(Y(u),u)$. By Theorem 1, there exists a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subseteq U_i = \mathbb{R}^Z$ such that $Y(u) \subseteq SB(Y(u),u) = PB(Y(u),u)$ for all $u \in \Pi_{i\in N}(U_i\backslash V^0)$. Therefore, for all $u \in \Pi_{i\in N}(U_i\backslash V^0)$, sequentially rationalizable sets are precisely perfectly rationalizable sets in $\Gamma(u)$. Now, suppose that $SB(Y(\cdot),\cdot)$ is lower hemi-continuous and $PB(Y(\cdot),\cdot)$ is upper hemi-continuous at u. By Theorem 1, $\{u \in U : SB(Y(u),u) \neq PB(Y(u),u)\} \subseteq U\backslash[\Pi_{i\in N}(U_i\backslash V^0)]$ is a lower dimensional subset of U. By Lemma 2, $SB(Y(u),u) \subseteq PB(Y(u),u)$. Therefore, $Y(u) \subseteq SB(Y(u),u) \subseteq PB(Y(u),u)$ is a perfectly rationalizable set.

Since $PR(u) = \cup_{Y\subseteq PB(Y,u)}Y$ and $SR(u) = \cup_{Y\subseteq SB(Y,u)}Y$, it follows that $SR(u) = PR(u)$ for all $u \in \Pi_{i\in N}(U_i\backslash V^0)$. Since $WPE(u) = \cup_{Y\subseteq PB(Y,u);\ |Y|=1}Y$ and $WSE(u) = \cup_{Y\subseteq SB(Y,u);\ |Y|=1}Y$ (where $|Y| = 1$ means that the cardinality of Y is 1), it follows that

$WPE(u) = WSE(u)$ for all $u \in \Pi_{i\in N}(U_i \backslash V^0)$. The rest of Corollary 1 follows immediately from Lemma 2. $\square$

Proof of Corollary 2. Let $\mathfrak{X} \equiv \{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^i x = {}^j x \text{ for all } i \neq j\}$. Then, $y \in SE(u)$ iff $y \in SB^{\mathfrak{X}}(y,u)$. Since $int(\mathbb{Y})$ is semi-algebraic in $\mathbb{Y}$, $\mathfrak{X}$ is semi-algebraic in $\mathbb{Y}^n$. By Theorem 1, we can find a semi-algebraic lower-dimensional subset U^0, such that $y \in PB^{\mathfrak{X}}(y,u) = SB^{\mathfrak{X}}(y,u)$ for all $u \in U \backslash U^0$. Note that, for all $u \in U \backslash U^0$, $PB^{\mathfrak{X}}(y,u) = SB^{\mathfrak{X}}(y,u)\ \forall y \in \mathbb{Y}$. Therefore, $PE(u) = \{y : y \in PB^{\mathfrak{X}}(y,u)\} = \{y : y \in SB^{\mathfrak{X}}(y,u)\} = SE(u)$ for all $u \in U \backslash U^0$.

Now, suppose that $SE(\cdot)$ is lower hemi-continuous and $PE(\cdot)$ is upper hemi-continuous at u. Since $\{u \in U : SE(u) \neq PE(u)\} \subseteq U^0$ is a lower dimensional subset, by Lemma 2, $SE(u) \subseteq PE(u)$. Thus, $SE(u) = PE(u)$. $\square$

Proof of Corollary 3. Consider a normal form $\Gamma = (N, \{A_i\}_{i\in N})$. Let $\left(W^k(u)\right)_{k=0}^K$ be an arbitrary (finite) IEWDS procedure in $\Gamma(u)$. Since Γ is a normal form, $a \in \mathcal{A}$ is not strictly dominated in $\mathcal{A}$ iff $a \in SB^{\mathfrak{X}(\mathcal{A})}(\Delta(\mathcal{A}), u)$; $a \in \mathcal{A}$ is not weakly dominated iff $a \in PB^{\mathfrak{X}(\mathcal{A})}(\Delta(\mathcal{A}), u)$, where $\mathcal{A} = \Pi_{i\in N}\mathcal{A}_i \subseteq \Pi_{i\in N} A_i$ and $\mathfrak{X}(\mathcal{A}) = [int(\Delta(\mathcal{A}))]^n$. Note that $\mathfrak{X}(\mathcal{A})$ is semi-algebraic and Theorem 1 holds true for all (finitely many) $\mathcal{A}$. Therefore, we can find a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that for all $u \in \Pi_{i\in N}(U_i \backslash V^0)$ and $k = 0, 1, ..., K-1$, $a \in W^k(u) \backslash W^{k+1}(u)$ iff a is strictly dominated in $W^k(u)$. That is, $\left(W^k(u)\right)_{k=0}^K$ is an IESDS procedure in $\Gamma(u)$. Since IESDS is order-independent, IEWDS is generically an order-independent procedure. $\square$

Proof of Corollary 4. By Theorem 1, there exists a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that, for all $u \in \Pi_{i\in N}(U_i \backslash V^0)$, $SB(Y,u) = PB(Y,u)\ \forall Y \subseteq \mathbb{Y}$. Suppose that a sequential-MACA $\sigma(u)$ is supported by Y. Then, for all $u \in \Pi_{i\in N}(U_i \backslash V^0)$, $Y \subseteq SB(co^e(Y), u) = PB(co^e(Y), u)$. Therefore, $\sigma(u)$ is also a perfect-MACA supported by Y. Since every perfect-MACA is a sequential-MACA, we conclude the proof. $\square$

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