A Unified Approach to Iterated Elimination Procedures*

Xiao Luo[a]†, Xuewen Qian[a], Chen Qu[b]

[a] *Department of Economics, National University of Singapore, Singapore 117570*

[b] *Department of Economics, BI Norwegian Business School, 0442 Oslo*

This version: June 2016

Abstract

We study the existence and uniqueness (or order independence) of iterated elimination procedure from a choice-theoretic viewpoint. We show a general existence result of iterated elimination procedure on an abstract reduction system. We identify a sufficient condition of "Monotonicity*" for the order independence and, in (in)finite games, we provide a full characterization of Monotonicity*. We also demonstrate that our approach is applicable to any form of iterated elimination processes in arbitrary strategic games, e.g., iterated strict dominance, iterated weak dominance, rationalizability, etc. *JEL Classification:* C70, D70.

Keywords: Iterated elimination procedures; order independence; Monotonicity*; Hereditarity*; CD* games

*Financial supports from National University of Singapore and BI Norwegian Business School are gratefully acknowledged. Acknowledgment to be added.

†Corresponding author. Fax:+65 6775 2646. *E-mail:* ecslx@nus.edu.sg (X. Luo), a0049466@u.nus.edu (X. Qian), chen.qu@bi.no (C. Qu).

1 Introduction

It is important and useful to define solution concepts by using iterated elimination procedures in game theory and economics. Notably, iterated elimination of strictly dominated strategies (IESDS), iterated elimination of weakly dominated strategies (IEWDS), iterated elimination of never-best responses (IENBR), and the backward induction principle are extensively discussed in game theory; see, e.g., Osborne and Rubinstein (1994, Chapter 4).[1] Intuitively, the iterated elimination procedures are profoundly related to the assumption of "common knowledge of rationality".

Most of the research in the literature has been restricted to specific forms of iterated elimination procedures in finite or CC games (where strategy sets are compact and payoff functions are continuous). We here consider any form of iterated elimination procedures in arbitrary strategic games. In this paper, we study the existence and uniqueness of final outcomes (or order independence) of iterated elimination procedure from a choice-theoretic viewpoint.[2] More specifically, we adopt the classical theory of choice in which the set of outcomes is formalized by a choice rule that specifies the acceptable/desirable choices. The reduction relation specifies that any feasible reduction from a given set is a deletion of elements outside the choice set. We consider all finite and transfinite sequences of reduction in any arbitrary abstract reduction system. For example, iterated deletion of strictly dominated strategies can be viewed as an (in)finite sequence of reduction in an abstract reduction system associated with the strict domination relation.

We show the existence of iterated elimination procedure for any arbitrary abstract reduction system (Theorem 1(a)). Except for the ZF axioms of set theory, our proof of the existence requires neither the Axiom of Choice nor the Well-Ordering Principle. Our existence theorem implies that there always exists an iterative elimination procedure for any arbitrary game by allowing a transfinite reduction sequence. In addition, under a (strong) condition of Monotonicity, the iterated elimination procedure is order independent and preserves all "fixed-points" (Theorem 1(b)).

In this paper, we follow Gilboa et al. (1990) to seek weaker sufficient conditions for order independence that can be used for various forms of iterated elimination procedure including finite and infinite elimination processes used in game theory. The major feature of this paper is that we impose no restrictions on the structure of games, possibly with infinite strategy spaces and discontinuous payoff functions. In the literature on game theory, most of the discussions on order independence focused on finite reduction sequences (in finite

[1] In contrast to the fixed-point method used in the equilibrium approach, this alternative approach develops solution concepts by using iterative procedures, for example, Bernheim (1984) and Pearce's (1984) notion of rationalizability, Dekel and Fudenberg's (1990) iterative procedure, Borgers's (1994) iterated pure-strategy dominance, Gul's (1996) τ-theories, Asheim and Dufwenberg's (2003) concept of a fully permissible set, Ambrus's (2006) definition of coalitional rationalizability, Cubitt and Sugden's (2011) reasoning-based iterative procedure, Halpern and Pass's (2012) iterated regret-minimization procedure, and Hillas and Samet's (2014) iterative elimination of flaws of weakly dominated strategies.

[2] Duggan and Le Breton (2014) modeled a player's decision as a choice set and analyzed set-valued solution concepts in finite games. Trost (2014) formulated each player's decision as an individual choice problem under uncertainty and offered some epistemic motivation for order-independent elimination procedures in finite games.

games); see, e.g., Gilboa et al. (1990), Osborne and Rubinstein (1994), Marx and Swinkels (1997), Asheim and Dufwenberg (2003), Apt (2004, 2011), Ambrus (2006, 2009), Tercieux (2006), Oyama and Tercieux (2009), Cubitt and Sugden (2011), Chen and Micali (2013) and Hillas and Samet (2014). Only a few of the research papers, e.g., Lipman (1994), Ritzberger (2002), Dufwenberg and Stegeman (2002), Green (2011), Chen et al. (2007, 2015), dealt with order independence for infinite reduction sequences in infinite games, with restrictions to the iterated strict dominance or rationalizability.[3] In this paper, we identify a fairly weak condition of "Monotonicity*" for the order independence on any arbitrary abstract reduction system (Theorem 2).

Roughly speaking, Monotonicity* requires that: along a reduction sequence, no undesirable alternative (which is outside a choice set) be changed to a desirable alternative after removing some of the undesirable alternatives, that is, choice sets are never expansive along an elimination path. The following example shows that, while iterated weak dominance is, in general, not an order-independent elimination procedure, it can be order independent and satisfy Monotonicity* in some particular game. Consider a two-person game (where player 1 chooses a row and player 2 chooses a column):

	y_1	y_2	y_3
x_1	$1, 1$	$1, 1$	$0, 0$
x_2	$0, 1$	$1, 1$	$2, 1$
x_3	$0, 0$	$0, 0$	$2, 1$

For any subset X of strategy profiles, choice set $c(X)$ consists of all weakly undominated strategy profiles in the reduced game with strategy-profile space X. The iterated weak dominance yields a unique outcome path: $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_3\} \to \{x_1, x_2\} \times \{y_1, y_2, y_3\} \to \{x_1, x_2\} \times \{y_1, y_2\} \to \{x_1\} \times \{y_1, y_2\}$, *along which the choice rule c satisfies a monotonicity property: $c(Y) \subseteq c(X)$ if $Y \subseteq X$.* (In this game, the choice rule c fails to satisfy the monotonicity property off this outcome path, e.g., $c(\{x_2\} \times \{y_2, y_3\}) \nsubseteq c(\{x_2, x_3\} \times \{y_2, y_3\})$.) That is, the game of this example satisfies the sufficient condition of Monotonicity* for order independence.[4]

In finite games, iterated strict dominance is indeed an order-independent elimination procedure, which actually satisfies Monotonicity* (because each strictly dominated strategy in any finite game remains to be strictly dominated in a reduced game after eliminating some of the strictly dominated strategies). However, in infinite games, iterated strict dominance might not be order-independent; the order dependence problem in infinite games is much more complicated and deeper (see Dufwenberg and Stegeman (2002) for extensive discussions). In particular, Monotonicity* may fail to be satisfied in this case: a strictly dominated strategy in an infinite game can be changed to a strictly undominated strategy

[3] See also Arieli (2010) and Halpern and Pass (2012) for related discussions on (infinitely) iterative elimination procedures.

[4] This example fails to satisfy the original "nice weak dominance" condition due to Marx and Swinkels (1997); see Section 3.1 for more discussions.

after eliminating some of the strictly dominated strategies.[5] Our main result of Theorem 2 implies that, if Monotonicity* holds, iterated strict dominance must be order independent in both finite games and infinite games. Exploring sufficient conditions for order independence for any kind of finitely and transfinitely iterated elimination procedures is the main focus of this paper.

We also apply our analysis to game theory. In finite games, Apt (2011) offered a uniform proof of order independence for various strategy elimination procedures based on Newman's (1942) Lemma; see also Apt (2004). We obtain Apt's (2011) Theorem 1 as a corollary of Theorem 2 (Corollary 1). In addition, we demonstrate how to apply our analysis of order independence to some of the iterated elimination processes discussed in the literature, including iterated strict dominance, iterated weak dominance and rationalizability.

In infinite games, we provide a full characterization of Monotonicity* by Hereditarity* (Theorem 3). In contrast to the Monotonicity* property on choice sets of desirable alternatives, Hereditarity* is a property for sets of undesirable alternatives – i.e. dominated elements under an abstract dominance relation – which can be often used in the context of games. Along the lines of Jackson's (1992) idea of "boundedness" which requires any eliminated strategy to be justified by an undominated dominator, we introduce a novel and useful definition of "closed under dominance* (CD*)" games, including all compact and own-uppersemicontinuous games, to establish an order independence result in infinite games. In CD* games, we show that Gilboa et al.'s (1990) procedure is an order-independent iterated elimination procedure (Corollary 4). In the special case of finite games, we also show that the result holds true under a simple form of 1-CD* games (Corollaries 2 and 3).

The rest of the paper is organized as follows. In Section 2, we define the iterated elimination procedure on an abstract reduction system and establish its existence. We investigate the uniqueness of iterated elimination procedure and show the order independence result under Monotonicity*. In Section 3, we apply our analysis to finite and infinite games. We provide a full characterization of Monotonicity* by Hereditarity*. We also show an order independence result in the class of CD* games. Section 4 concludes. To facilitate reading, all the proofs are relegated to the appendixes.

2 Iterated Elimination Procedures

Consider an arbitrary set S of alternatives.[6] A *choice rule on* S is a function $c : 2^S \to 2^S$ which designates a *choice set* $c(X) \subseteq X$ for each subset $X \subseteq S$. For the purpose of this paper, we do not require the nonemptiness of choice sets. (Note that, for arbitrary function $f : 2^S \to 2^S$, we can define a choice rule c_f on S by $c_f(X) = X \cap f(X)$ for all $X \subseteq S$.) We interpret that, when faced with the set X of alternatives, all elements in the choice set $c(X)$ are regarded as "choosable/acceptable" outcomes – the alternatives that can be

[5] For example, consider a simple one-person game where the strategy space is $X = (0, 1)$ and the payoff function is $u(x) = x$ for every strategy x. Obviously, every strategy is strictly dominated and choice set $c(X) = \varnothing$. We can eliminate in round one all strategies except a particular strategy x in $(0, 1)$. Thus, $c(\{x\}) = \{x\} \nsubseteq c(X)$, which violates Monotonicity*.

[6] Throughout this paper, we assume that sets satisfy the ZF axioms (cf., e.g., Jech 2003, p.3).

chosen; cf. Sen (1993, p.499). Throughout this paper, we denote by X and Y subsets of S. A choice rule c is said to satisfy *Monotonicity* if

$$[Y \subseteq X] \Rightarrow [c(Y) \subseteq c(X)];$$

that is, there are no fewer acceptable outcomes available within a wider scope of feasible alternatives.

We define the *reduction relation* $\rightarrow$ for the choice problem (S, c) as follows:

$$X \rightarrow Y \text{ iff } c(X) \subseteq Y \subseteq X.$$

That is, X can be reduced to Y iff no element in $c(X)$ is eliminated from X to a subset Y of X. Apparently, we allow $X \rightarrow X$ for any $X \subseteq S$. We denote by $(S, \rightarrow)$ the *abstract reduction system* for the choice problem (S, c). We define the iterated elimination procedure on the abstract reduction system $(S, \rightarrow)$, possibly by using a transfinite process of reduction.[7] Let 0 denote the first element in an ordinal Λ, and let $\lambda + 1$ denote the successor to λ in Λ.

Definition 1. An *iterated elimination process (IEP) for the choice problem* (S, c) is a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$ on $(S, \rightarrow)$ such that

(a) $X^0 = S$,

(b) $X^\lambda \rightarrow X^{\lambda+1}$ (and $X^\lambda = \cap_{\lambda' < \lambda} X^{\lambda'}$ for a limit ordinal λ), and

(c) $X^\Lambda \rightarrow X$ only if $X^\Lambda = X$.

In Definition 1(c), the "stopping" condition: $X^\Lambda \rightarrow X$ only if $X^\Lambda = X$ expresses the idea that no elements in X^Λ can be eliminated for further consideration; it is equivalent to $X^\Lambda = c(X^\Lambda)$. An IEP $\{X^\lambda\}_{\lambda \leq \Lambda}$ for (S, c) is "fast" if $X^{\lambda+1} = c(X^\lambda)$ and $X^\Lambda \neq X^\lambda$ for all $\lambda < \Lambda$. Definition 1 does not require the elimination of *all* elements outside the choice set $c(X^\lambda)$ in each round of reduction since $c(X^\lambda) \subseteq X^{\lambda+1} \subseteq X^\lambda$; in particular, it allows for the elimination of no elements in some round of reduction: $X^{\lambda+1} = X^\lambda$.

The following theorem asserts that, for any arbitrary set S and choice rule c on S, there is always an iterated elimination process in Definition 1. Under Monotonicity, the iterated elimination procedure is *order-independent*: that is, all IEPs yield a unique set of final outcomes. Furthermore, every IEP results in all "fixed-points" of c and, thus, it preserves all elements $x = c(x)$.

Theorem 1. *(a) For any problem (S, c), there exists a (fast) IEP on $(S, \rightarrow)$. (b) Suppose that c satisfies Monotonicity. Then, the iterated elimination procedure is order-independent; moreover, if $\{X^\lambda\}_{\lambda \leq \Lambda}$ is an IEP for (S, c), then $X^\Lambda = \cup_{Z = c(Z)} Z$.*

[7] Since the set S may be infinite, it is natural and necessary for us to consider a transfinite sequence of reduction on $(S, \rightarrow)$. Lipman (1994) demonstrated that, in infinite games, we need the transfinite induction to deal with the strategic implication of "common knowledge of rationality"; see also Chen et al. (2007, Example 1) and Green (2011) for more discussions.

We would like to point out that, except for the ZF axioms, our proof of the existence of iterated elimination procedure does not require the Axiom of Choice. The proof improves, if applied to iterated strict dominance in games, the existence proofs in Chen et al. (2007, 2015) which rely on either the Axiom of Choice or the Well-Ordering Principle.

The iterated elimination procedure is in general order-dependent: iterated elimination processes in Definition 1 may generate different sets of outcomes. For instance, some of the most prominent iterated elimination procedures such as iterated elimination of weakly dominated strategies (IEWDS) fail to be order independent. Under Monotonicity, Theorem 1(b) asserts that the iterated elimination procedure must be order-independent. While this result is simple, it is useful to determine the order independence for many iterated elimination processes used in game theory, which preserve Nash equilibria by Theorem 1(b). See, for example, Apt's (2004) related discussions on the order independence of various forms of iterated dominance in finite games, Ritzberger's (2002) Theorem 5.1 for the order independence of iterated strict dominance in the class of CC games (where strategy sets are compact and payoff functions are continuous), and Chen et al.'s (2007) Theorem 1 for the order independence of iterated strict dominance* in arbitrary games.

Nevertheless, iterated elimination of strictly dominated strategies (IESDS) fails to satisfy the property of monotonicity (because no element in a singleton of a strictly dominated strategy can be strictly dominated by using this dominated strategy). Monotonicity is not a necessary condition for order independence, as illustrated by the example in Introduction. In particular, the monotonicity property seems to be an unnecessary requirement for a circumstance that never occurs in performing the iterated elimination. We offer a weaker version of monotonicity for order independence, which we call "monotonicity*": it requires the monotonicity property *only along the iterated reduction sequence starting at* S. Let $\rightarrow^*$ denote the *indirect reduction relation* induced by $\rightarrow$. That is, $X \rightarrow^* Y$ iff there is a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$ such that $X^0 = X$, $X^\lambda \rightarrow X^{\lambda+1}$ (and $X^\lambda = \cap_{\lambda' < \lambda} X^{\lambda'}$ for a limit ordinal λ), and $X^\Lambda = Y$. In other words, $X \rightarrow^* Y$ via a reduction sequence $\left\{X^\lambda\right\}_{\lambda \leq \Lambda}$.

Monotonicity*. $[S \rightarrow^* X \rightarrow^* Y] \Rightarrow [c(Y) \subseteq c(X)]$.

That is, the Monotonicity* property requires that, along a reduction sequence through X to Y, choice sets should not be expansive – i.e., no undesirable alternative outside $c(X)$ can be changed to a desirable one in $c(Y)$ during the phase of reduction: $X \rightarrow^* Y$. If we restrict our attention only to the one-step-ahead reduction, we obtain a simpler and weaker version of *1-Monotonicity**: $[S \rightarrow^* X \rightarrow Y] \Rightarrow [c(Y) \subseteq c(X)]$. The central result of this paper is that, under the weak condition of Monotonicity*, the iterated elimination procedure is order-independent. Moreover, 1-Monotonicity* is sufficient for order independence in the finite case.

Theorem 2. *(a) Suppose that* c *satisfies Monotonicity*. Then, the iterated elimination procedure for the problem* (S, c) *is order-independent. (b) Let* S *be a finite set. If* c *satisfies 1-Monotonicity*, then* c *satisfies Monotonicity* and, thus, the iterated elimination procedure is order-independent.*

Remark. Monotonicity implies Monotonicity* which in turn implies 1-Monotonicity*. 1-Monotonicity* is closely related to the Aizerman property used in the choice-theoretic literature: $[c(X) \subseteq Y \subseteq X] \Rightarrow [c(Y) \subseteq c(X)]$; see, e.g., Moulin (1985). In 1-Monotonicity*, we relax the premise of the Aizerman property only for the set X resulting from a reduction sequence starting at S. In Monotonicity*, we strengthen the premise of the Aizerman property by considering the indirect reduction relation $\to^*$ because the iterated elimination procedure may use a transfinite sequence of reduction in infinite sets.

The following example shows that, in the infinite set case, 1-Monotonicity* is not sufficient for order independence.

Example 1. Consider an infinite set $S = \mathbb{N} \cup \{-1\}$. The choice rule c is defined as: for subsets $N \subseteq \mathbb{N}$,

$$c(N \cup \Psi) = \begin{cases} N \backslash \min N & \text{if } |N| > 1 \\ N & \text{if } |N| = 1 \\ \Psi & \text{if } N = \varnothing \end{cases},$$

where Ψ is $\{-1\}$ or $\varnothing$. In this example there are two IEPs which generate different outcomes:

1. $X^0 = S$, $X^n = \mathbb{N} \backslash \{0, 1, ..., n-1\}$ for all $n \in \mathbb{N}$, and $X^{\mathbb{N}} = \cap_{n \in \mathbb{N}} X^n = \varnothing$.
2. $\widetilde{X}^0 = S$, $\widetilde{X}^n = S \backslash \{0, 1, ..., n-1\}$ for all $n \in \mathbb{N}$, and $\widetilde{X}^{\mathbb{N}} = \cap_{n \in \mathbb{N}} \widetilde{X}^n = \{-1\}$.

The choice rule c satisfies 1-Monotonicity*. However, c fails to satisfy Monotonicity* since $S \to^* \{-1\}$ and $\{-1\} = c(\{-1\}) \nsubseteq c(S) = \mathbb{N} \backslash \{0\}$. (Note: $\cap_{n \in \mathbb{N}} c\left(\widetilde{X}^n\right) \neq c\left(\cap_{n \in \mathbb{N}} \widetilde{X}^n\right)$.)

3 Applications in game theory

Consider an arbitrary (strategic) game:

$$G \equiv (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}),$$

where N is an (in)finite set of players, S_i is an (in)finite set of player i's strategies, and $u_i \colon \times_{i \in N} S_i \to \mathbb{R}$ is player i's arbitrary payoff function. Let $S \equiv \times_{i \in N} S_i = S_i \times S_{-i}$. For $X \subseteq S$ let $X_i = \{s_i \in S_i : \ (s_i, x_{-i}) \in X\}$ and $X_{-i} = \{s_{-i} \in S_{-i} : \ (x_i, s_{-i}) \in X\}$. For game G, let c be a choice rule on S and for $X \subseteq S$ define

$$DOM(X) \equiv X \backslash c(X).$$

We consider the abstract reduction system $(S, \to)$ for the choice problem (S, c).

3.1 Finite games

3.1.1 Hereditarity

Consider a finite game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$. In finite games, along the lines of Gilboa et al.'s (1990) approach, Apt (2011) presented an easy-to-apply condition of "hereditarity" for order independence; Apt (2011) demonstrated that many of order-independence results for iterated elimination procedures in finite games can be obtained by checking the hereditarity property. We state this condition in our choice-based setting as follows: Let $X, Y \subseteq S$.

HEREDITARITY. $[X \to Y] \Rightarrow [(Y \cap DOM(X)) \subseteq DOM(Y)]$.

That is, Hereditarity says if x is dominated but not eliminated (i.e., $x \in DOM(X) \cap Y$), then it is still dominated after eliminating some of the dominated elements (i.e., $x \in DOM(Y)$). For example, under the strict domination relation, a finite game satisfies Hereditarity because every strictly dominated strategy in a finite game has an undominated dominator, remaining in any reduced game after eliminating some of the strictly dominated strategies, which strictly dominates the former dominated strategy in this reduced game. Since Hereditarity actually implies 1-Monotonicity* (see the proof of Corollary 1), Apt's (2011, Theorem 1) order-independence result follows immediately from Theorem 2(b).[8]

Corollary 1. *Suppose that G is a finite game. Under Hereditarity, the iterated elimination procedure is order-independent.*

3.1.2 1-CD* games

Motivated by Jackson's (1992) idea of "boundedness" that requires that strategies be eliminated only by undominated strategies, Dufwenberg and Stegeman (2002) introduced a definition of "games closed under dominance (CD games)" for the strict dominance and showed, through an example, that IESDS can be order-dependent in CD games. Roughly speaking, CD games satisfy the property that at any point in a finite-step sequence of deletions, any dominated strategy has an undominated dominator. At a conceptual level, this definition of CD games does not rule out the possibility of order dependence of an elimination procedure. The following example shows this point.[9]

Example 2. Consider the two-person game:

	y_1	y_2
x_1	$1, 1$	$0, 0$
x_2	$1, 1$	$2, 1$

[8] Apt (2011) considered the class of finite sequences of reduction under a variety of dominance relations in finite games and showed this result by using Newman's (1942) Lemma.

[9] By the transitivity of weak dominance, any finite game is a CD game; see Jackson (1992, p.763). This definition of CD games cannot be expected to solve the problem of order dependence in nature.

The game is a CD game under the weak dominance because, in every reduced game, any weakly dominated strategy is weakly dominated by a weakly undominated strategy. However, the iterated weak dominance is not order independent. At a conceptual level, the order independence of iterated strict dominance should not be attributed to the CD property, but due to the fact that Hereditarity holds for iterated strict dominance in finite games.

Nevertheless, the strict domination relation in finite games satisfies a stronger 1-CD* property that every strictly dominated strategy in a finite game has an undominated dominator, remaining to be an undominated dominator in any reduced game after eliminating some of the strictly dominated strategies, which strictly dominates the former dominated strategy in this reduced game. In fact, if an abstract domination relation satisfies Monotonicity like ones under rationalizability and the strict dominance* as defined below, then CD implies this stronger 1-CD* property, which is sufficient for order independence of the iterated elimination procedure defined by using reduced games because it implies Hereditarity.

We follow Jackson's (1992) idea of "boundedness" to introduce the notion of 1-CD* games to solve the problem of order dependence under an arbitrary domination relation. Consider a finite game $G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$. For $X \subseteq S = \times_{i\in N} S_i$, let $\succ_X$ denote an *abstract domination relation* on S given X. For profiles $x, y \in S$, $y \succ_X x$ is interpreted to mean that y dominates x conditionally on X; see also Luo (2001) for more discussions. For instance, $y \succ_X x$ can represent the strict domination relation: "y strictly dominates x given X", that is, $y \succ_X x$ iff there exists $i \in N$ such that $u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$ for all $z_{-i} \in X_{-i}$. Define $c(X) = X \backslash DOM(X)$, where

$$DOM(X) = \{x \in X : y \succ_X x \text{ for some } y \in X\}.$$

For any $X, Y \subseteq S$, let

$$DOM^Y(X) = \{x \in X : \ y \succ_X x \text{ for some } y \in Y\}.$$

Now consider the abstract reduction system $(S, \rightarrow)$ for the problem (S, c). We say that game G (under an abstract domination relation $\succ_X$) is *one-step-ahead-deletion closed under dominance* (1-CD*)* if $S \rightarrow^* X \rightarrow Y$ and $y \succ_X x$ for some $y \in X$ and $x \in Y$ imply that: there exists $z^* \in Y$ such that $z \not\succ_Y z^* \succ_Y x$ for all $z \in Y$, i.e., $[S \rightarrow^* X \rightarrow Y] \Rightarrow \left[(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y)\right]$. That is, at any stage in a sequence of deletions, any dominated element surviving the one-step-ahead deletion has an undominated dominator which continues dominating it at the end of the deletion. In other words, under an abstract domination relation, any dominated strategy that is not eliminated has an undominated dominator during the phase transition of one-step-ahead deletion.[10]

[10]That is, a 1-CD* game has a "boundedness" property that there exists the undominated dominator which is not eliminated in the one-step-ahead deletion. This property is related to Gilboa, Kalai, and Zemel's (1990) (GKZ) notion of reduction that requires that the dominator is one which is not eliminated. Corollary 3 shows that, in 1-CD* games, the GKZ elimination procedure is equivalent to the reduction procedure discussed in this paper.

Note that a 1-CD* game must be a CD game because the former also satisfies the property: $S \to^* X \to X$ and $y \succ_X x$ for some $x, y \in X$ imply that: there exists $z^* \in X$ such that $z \not\succ_X z^* \succ_X x$ for all $z \in X$, i.e., $[S \to^* X] \Rightarrow \left[DOM(X) = DOM^{c(X)}(X)\right]$. Thus, the notion of 1-CD* games can be viewed as a "dynamic" version of CD games by extending the CD concept to the one-step-ahead reduction transition. The notion of 1-CD* games can exclude problematic games with the problem of order dependence; for instance, the game of Example 2 is not a 1-CD* game under the weak domination relation since $S \to X = \{x_1,\ x_2\} \times \{y_1\}$ and $(x_2, y_1) \succ_S (x_1, y_1)$ but $(x_2, y_1) \not\succ_X (x_1, y_1)$. The following corollary asserts that the iterated elimination procedure is order independent for any finite 1-CD* game.

Corollary 2. *Suppose that G is a finite 1-CD* game. Then, the iterated elimination procedure is order-independent.*

Gilboa, Kalai and Zemel (1990) (GKZ) studied a variety of elimination procedures and provide sufficient conditions for order independence. The key requirement for the GKZ procedure is that for any element x that is eliminated there exists an element y that dominates x and is not eliminated. More precisely, the GKZ procedure is an elimination procedure on the abstract reduction system $\left(S, \to^{\text{GKZ}}\right)$ associated with an abstract domination relation $\succ_X$, where, for subsets $X, Y \subseteq S$,

$$X \to^{\text{GKZ}} Y \text{ iff } Y \subseteq X \text{ and } X \backslash Y \subseteq DOM^Y(X).$$

That is, $X \to^{\text{GKZ}} Y$ iff every eliminated element $x \in X \backslash Y$ has a dominator $y \in Y$ (i.e. $x \in DOM^Y(X)$). Apparently, $X \to^{\text{GKZ}} Y$ implies $X \to Y$, since $DOM^Y(X) \subseteq DOM(X)$; the GKZ procedure can be viewed as a special form of the iterated elimination procedure in Definition 1. Moreover, if $DOM(X) \subseteq DOM^{c(X)}(X)$, $X \to^{\text{GKZ}} Y$ iff $X \to Y$. The following Corollary states that, in finite 1-CD* games, the iterated elimination procedure in Definition 1 is precisely the GKZ procedure, which is an order-independent procedure as proved by GKZ.

Corollary 3. *Suppose that G is a finite 1-CD* game. Then, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1 and, thus, the GKZ procedure is order independent.*

3.1.3 Related literature

We demonstrate how to apply our analysis of order independence to some of the iterated elimination processes for finite games discussed in the literature, including iterated strict dominance, iterated weak dominance and rationalizability. For any subset X of strategy profiles, we can define the choice set $c(X)$ in the following ways.

1. **[strict dominance]** $c(X) = X \backslash DOM(X)$ where[11]

$$DOM(X) = \left\{ x \in X : \ \exists i \in N \ \exists \sigma_i \in \Delta(X_i) \text{ s.t. } u_i\left(\sigma_i, x'_{-i}\right) > u_i\left(x_i, x'_{-i}\right) \ \forall x'_{-i} \in X_{-i} \right\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is strictly dominated by no mixed strategy in $\Delta(X_i)$. Since every strictly dominated strategy x_i in a finite game has an undominated dominator, remaining in a reduced game after eliminating some of the strictly dominated strategies, which strictly dominates x_i in that reduced game, $(Y \cap DOM(X)) \subseteq DOM(Y)$ for $c(X) \subseteq Y \subseteq X$. Thus, Hereditarity holds. By Corollary 1, IESDS is an order-independent procedure. (Under the strict dominance relation, 1-CD* actually holds true, but Monotonicity fails to be satisfied, e.g., $c(x) = x \notin c(X)$ for $x \in X \backslash c(X)$.)

2. **[weak dominance]** $c(X) = X \backslash DOM(X)$ where

$$DOM(X) = \left\{ x \in X : \begin{array}{c} \exists i \in N \ \exists \sigma_i \in \Delta(X_i) \text{ s.t. } u_i\left(\sigma_i, x'_{-i}\right) \geq u_i\left(x_i, x'_{-i}\right) \ \forall x'_{-i} \in X_{-i} \\ \text{and } u_i\left(\sigma_i, x'_{-i}\right) > u_i\left(x_i, x'_{-i}\right) \text{ for some } x'_{-i} \in X_{-i} \end{array} \right\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is weakly dominated by no mixed strategy in $\Delta(X_i)$. The IEWDS procedure may not be order independent in general; see, e.g., Example 2.

3. **[strict dominance*]** $c(X) = X \backslash DOM^S(X)$ where

$$DOM^S(X) = \left\{ x \in X : \ \exists i \in N \ \exists s_i^* \in S_i \text{ s.t. } u_i\left(s_i^*, x'_{-i}\right) > u_i\left(x_i, x'_{-i}\right) \ \forall x'_{-i} \in X_{-i} \right\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is strictly dominated by no strategy in S_i; see, e.g., Milgrom and Roberts (1990), Ritzberger (2002) and Chen et al. (2007). Since every strictly dominated strategy in a finite game has an undominated dominator which strictly dominates that dominated strategy in each of subgames, $\left(Y \cap DOM^S(X)\right) \subseteq DOM^S(Y)$ for $Y \subseteq X$. Thus, $c(Y) \subseteq c(X)$ if $Y \subseteq X$. That is, Monotonicity holds. By Theorem 1(b), The IESDS* procedure is order independent and preserves Nash equilibria.

4. **[pure-strategy dominance]** $c(X) = X \backslash DOM(X)$ where

$$DOM(X) = \left\{ x \in X : \begin{array}{c} \exists i \in N \ \forall Z_{-i} \subseteq X_{-i} \ \exists s_i^* \in S_i \text{ s.t. } u_i(s_i^*, z_{-i}) \geq u_i(x_i, z_{-i}) \\ \forall z_{-i} \in Z_{-i} \text{ and } u_i(s_i^*, z_{-i}) > u_i(x_i, z_{-i}) \text{ for some } z_{-i} \in Z_{-i} \end{array} \right\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is undominated in the sense of Borgers (1994). Under the pure-strategy dominance relation,

[11] We denote by $\Delta(X_i)$ the probability space on X_i and by $u_i(\sigma_i, x_{-i})$ the expected payoff of player i under a mixed strategy $\sigma_i \in \Delta(X_{-i})$.

since every dominated strategy in a finite game is clearly dominated in each subgame, $(Y \cap DOM(X)) \subseteq DOM(Y)$ for $Y \subseteq X$. Thus, Hereditarity holds. By Corollary 1, the pure-strategy dominance is an order-independent reduced procedure.

5. **[rationalizability]** $c(X) = X \cap BR(X)$ where[12]

$$BR(X) = \{s \in S : \forall i \in N \ \exists \mu_i \in \Delta(X_{-i}) \text{ s.t. } u_i(s_i, \mu_i) \geq u_i(s'_i, \mu_i) \ \forall s'_i \in S_i\}.$$

   That is, $c(X)$ consists of all elements in X where each player i's strategy is a best response to some probabilistic belief in $\Delta(X_{-i})$. Since $BR(Y) \subseteq BR(X)$ for $Y \subseteq X$, $c(Y) \subseteq c(X)$ if $Y \subseteq X$. That is, Monotonicity holds. By Theorem 1(b), rationalizability is an order-independent elimination of never best response strategies which preserves Nash equilibria.

6. **[c-rationalizability]** Ambrus (2006) proposed a solution concept of "coalitional rationalizability (c-rationalizability)" in finite games by an iterative procedure of restrictions of strategies. The procedure is analogous to iterative elimination of never best response strategies, but operates on implicit agreements by coalitions. More specifically, let X and Z be product-form subsets of strategy profiles. Z is a *supported restriction* by coalition $J \subseteq N$ given X if (i) $Z_j \subseteq X_j$ for $j \in J$ and $Z_i = X_i$ for $i \notin J$ and (ii) for $j \in J$, $x_j \in X_j \backslash Z_j$ implies that

$$\max_{f_{-j} \in \Delta(X_{-j})} u_j(x_j, f_{-j}) < \max_{s_j \in S_j} u_j(s_j, g_{-j}) \ \forall g_{-j} \in \Delta(Z_{-j}) \text{ with } g_{-j}^{-J} = f_{-j}^{-J}$$

   where f_{-j}^{-J} and g_{-j}^{-J} are the marginal distributions of f_{-j} and g_{-j} over S_{-J} respectively. Let $\mathcal{F}(X)$ be the set of all the supported restrictions given X. We define the choice rule c for c-rationalizability by

$$c(X) = \cap_{Z \in \mathcal{F}(X)} Z.$$

   Ambrus (2006) defined c-rationalizability by a (fast) iterated elimination procedure associated with this choice rule c; that is, in each elimination round the intersection of all supported restrictions is retained (see also Ambrus (2009) and Luo and Yang (2012) for more discussions). Ambrus (2006, Proposition 5) showed an order independence result, *under the restriction that each elimination round must be an intersection of some supported restrictions.* Since the choice rule c satisfies 1-Monotonicity* (see Lemma 4 in Appendix), by Theorem 2(b), Ambrus's (2006) notion of c-rationalizability is an order-independent procedure, without the aforementioned restriction.

7. **[HS-weak dominance]** $c(X) = X \backslash DOM(X)$ where

$$DOM(X) = \left\{ x \in X : \begin{array}{c} \exists i \in N \ \exists \sigma_i \in \Delta(S_i) \text{ s.t. } u_i(\sigma_i, x_{-i}) > u_i(x) \\ \text{and } u_i(\sigma_i, x'_{-i}) \geq u_i(x_i, x'_{-i}) \ \forall x'_{-i} \in X_{-i} \end{array} \right\}.$$

[12] We denote by $\Delta(X_{-i})$ the probability space on X_{-i} and by $u_i(x_i, \mu_i)$ the expected payoff of player i under a probabilistic belief $\mu_i \in \Delta(X_{-i})$.

That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is not weakly undominated in the sense of Hillas and Samet (2014, Definition 4). Under the HS-weak dominance relation, since every dominated strategy in a finite game has an undominated dominator which dominates that dominated strategy in each of subgames, $(Y \cap DOM(X)) \subseteq DOM(Y)$ for $c(X) \subseteq Y \subseteq X$. That is, Hereditarity holds. By Corollary 1, the HS-weak dominance is an order-independent procedure; see Hillas and Samet's (2014) Proposition 2.

8. **[IECFA]** Asheim and Dufwenberg (2003) defined the concept of a "fully permissible set", which captures an idea of "common certain belief" that each player avoids using a weakly dominated strategy, by an "iterative elimination of choice sets under full admissible consistency (IECFA)" in finite games. In a two-player game, the iterated elimination procedure can be simply defined on $\Sigma = \Sigma_1 \times \Sigma_2$ (instead of S), where $\Sigma_i = 2^{S_i} \backslash \{\varnothing\}$. For any nonempty $\Xi = \Xi_1 \times \Xi_2 \subseteq \Sigma$, define $c(\Xi) = c_1(\Xi_2) \times c_2(\Xi_1)$, where

$$c_i(\Xi_j) \equiv \{Q_i \in \Sigma_i : \exists (\varnothing \neq) \mathcal{Q} \subseteq \Xi_j \text{ s.t. } Q_i = S_i \backslash D_i(\mathcal{Q})\} \text{ and}$$

$$D_i(\mathcal{Q}) \equiv \left\{s_i \in S_i : \exists x_i \in \Delta(S_i) \text{ weakly dominates } s_i \text{ on } S_j \text{ or } \cup_{Q_j \in \mathcal{Q}} Q_j\right\}.$$

As Asheim and Dufwenberg (2003) pointed out, Monotonicity holds for the choice rule c. By Theorem 1(b), IECFA is an order independent procedure.

9. **[RBEU]** Cubitt and Sugden (2011) offered an iterative procedure of "reasoning-based expected utility procedure (RBEU)" for solving finite games. RBEU uses a sequence of "accumulation" and "elimination" operations to categorize strategies as permissible and impermissible; some strategies remain uncategorized when the procedure halts. Cubitt and Sugden (2011) demonstrated that RBEU can delete more strategies than IESDS, while avoiding the order dependence problem associated with IEWDS. Formally, a sequence of categorizations $\{S(k)\}_{k=0}^{\infty}$ is inductively defined as: (i) $S(0) \equiv (\varnothing, \varnothing)$ and (ii) for all $k \geq 1$, $S(k) \equiv (S^+(k), S^-(k))$ such that, for all $i \in N$,

$$S_i^+(k) \subseteq \left\{s_i \in S_i : \ \forall \mu \in \Delta_{k-1}^*, \ u_i(s_i, \mu) \geq u_i(s_i', \mu) \text{ for all } s_i' \in S_i\right\};$$
$$S_i^-(k) \subseteq \left\{s_i \in S_i : \ \forall \mu \in \Delta_{k-1}^*, \ u_i(s_i', \mu) > u_i(s_i, \mu) \text{ for some } s_i' \in S_i\right\},$$

where $\Delta_{k-1}^* = \left\{\mu \in \Delta(S_{-i}) : \mu\left(S_{-i}^-(k-1)\right) = 0 \text{ and } \mu(s_{-i}) > 0 \ \forall s_{-i} \in S_{-i}^+(k-1)\right\}$. Given a sequence $\{S(k)\}_{k=0}^{\infty}$, we can alternatively define a class of IEPs under the choice rule $c(X) = X \backslash \cup_{k \in \left\{k' : \, X \subseteq S \backslash S_i^-(k')\right\}} S_i^-(k)$. Apparently, Monotonicity holds for this choice rule c and, by Theorem 1(b), it generates an order-independent reduced procedure, which leads to a unique set of final outcomes by Cubitt and Sugden's (2011) Proposition 2.

10. [**nice weak dominance**] Let X denote a product set of S (i.e., $X = \times_{i \in N} X_i \subseteq S$). We say that player i's strategy $s_i \in S_i$ is *nicely weakly dominated (NWD)* on X, if there exists $s_i^* \in S_i$ such that for all $x_{-i} \in X_{-i}$, either $u_i(s_i^*, x_{-i}) > u_i(s_i, x_{-i})$ or

$u(s_i^*, x_{-i}) = u(s_i, x_{-i})$; and the former inequality holds for some $x_{-i} \in X_{-i}$.[13] Define $c(X) \equiv \times_{i \in N} c_i(X) \equiv \times_{i \in N} (X_i \backslash DOM_i(X))$ where

$$DOM_i(X) = \{x_i \in X_i : x_i \text{ is NWD on } X \text{ by some } x_i^* \in X_i\}.$$

That is, $c_i(X)$ consists of all player i's strategies which are not nicely weakly dominated in the sense of Marx and Swinkels (1997, Definition 2). Marx and Swinkels (1997) showed that iterated elimination of nicely weakly dominated strategies is "outcome" order independent. This choice rule c for the nice weak dominance satisfies a variant of 1-Monotonicity* which gives rise to the desirable order-independence result (see Appendix II: NWD).

3.2 Infinite games

3.2.1 Hereditarity*: a full characterization

Consider an infinite game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$. It seems natural and necessary to account for both finite and transfinite sequences of deletions. We present a variant form of "hereditarity" for order independence in infinite games.

Hereditarity*. $[S \to^* X \to^* Y] \Rightarrow [(Y \cap DOM(X)) \subseteq DOM(Y)]$.

Hereditarity* is a property for the domination relation used in a game. The following result shows that the Hereditarity* property provides a full characterization for Monotonicity* in the context of a game. Therefore, Hereditarity* provides an alternative sufficient condition for order independence in infinite games.

Theorem 3. *Hereditarity* and Monotonicity* are equivalent. Under Hereditarity*, the iterated elimination procedure is order independent.*

3.2.2 CD* games

In infinite games, restricted to iterated elimination of strictly dominated strategies (IESDS), Dufwenberg and Stegeman (2002) wrote, "More surprising, ... requiring that strategies be eliminated only by undominated strategies (a variation on Jackson's (1992) idea of "boundedness") does not solve the problem of order dependence ... We concluded that the problems of IESDS in infinite games are deeper than the possible nonexistence of the "best" dominating strategy." The following example, taken from Dufwenberg and Stegeman (2002), shows that the GKZ procedure cannot solve the problems of order dependence in CD games.

[13] That is, s_i is weakly dominated by s_i^* and the game satisfies the "transference of decisionmaker indifference (TDI)" condition: whenever a player is indifferent between two profiles that differ only in the player's strategy, that indifference is transferred to the opponents as well.

Example 3. Consider a two-person game: $G \equiv (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$, where $N = \{1, 2\}$, $S_1 = S_2 = [0, 1] \backslash \{1/3\}$, and $u_i : S_i \times S_j \mapsto \mathbb{R}$ for $i, j \in N$ and $i \neq j$, defined by

$u_i(s_i, s_j) = s_i(1 - s_i - s_j)$ if $s_j \in \mathbb{Q}$,

$u_i(s_i, s_j) = s_i(1 - s_i - 1/3)$ if $s_j \notin \mathbb{Q}$,

where $\mathbb{Q}$ denotes the set of rational numbers in $[0, 1]$. Dufwenberg and Stegeman (2002, p.2017) showed that this game is a CD game. However, the IESDS procedure fails to be order independent; for example, there are two IESDS processes that generate different outcomes: Let $\langle a, b \rangle^2 \equiv [a, b] \backslash \{1/3\} \times [a, b] \backslash \{1/3\}$.

1. $X^0 = S = \langle a^0, b^0 \rangle^2$, $X^n = \langle a^n, b^n \rangle^2$ (where $a^n = (1 - b^{n-1})/2$ and $b^n = (1 - a^{n-1})/2$), and $X^{\mathbb{N}} = \cap_{n \in \mathbb{N}} X^n = \varnothing$.

2. $\widetilde{X}^0 = S$, $\widetilde{X}^n = X^n \cup \{(q, q)\}$ for $n \in \mathbb{N}$, and $\widetilde{X}^{\mathbb{N}} = \cap_{n \in \mathbb{N}} \widetilde{X}^n = \{(q, q)\}$ (where $q \notin \mathbb{Q}$).

Particularly, for all $n \in \mathbb{N}$, $c\left(\widetilde{X}^n\right) = X^{n+1}$. Thus, $\varnothing = \cap_{n \in \mathbb{N}} c\left(\widetilde{X}^n\right) \not\supseteq c\left(\widetilde{X}^{\mathbb{N}}\right) = \{(q, q)\}$. That is, this game lacks the continuity at the limit point of deletions; the choice rule c is explosive at the limit point. In Example 1 in Section 2, the choice rule c also displays "upward jumps" at the limit point: $\cap_{n \in \mathbb{N}} c\left(\widetilde{X}^n\right) \subset c\left(\widetilde{X}^{\mathbb{N}}\right)$. (Note: This game of Example 3 is a 1-CD* game.)

In order to get rid of the problem of order dependence in games with infinite strategy sets, we need to introduce a stronger notion of CD* games. We say that game G (under an abstract domination relation $\succ_X$) is *closed under dominance* (CD*)* if $S \rightarrow^* X \rightarrow^* Y$ and $y \succ_X x$ for some $y \in X$ and $x \in Y$ imply that: there exists $z^* \in Y$ such that $z \not\succ_Y z^* \succ_Y x$ for all $z \in Y$, i.e., $[S \rightarrow^* X \rightarrow^* Y] \Rightarrow \left[(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y)\right]$. That is, at any point in any valid sequence of deletions, any dominated element surviving the deletion process has an undominated dominator at the end point of the deletion which dominates it.

The following result asserts that, in the class of CD* games, there is no problem of order dependence. In particular, the GKZ procedure (by allowing an transfinite sequence of elimination) is always order independent. Under the strict dominance relation, all compact and own-uppersemicontinuous (COUSC) games are CD* and, hence, the IESDS procedure in Definition 1 is well-defined and order independent.

Corollary 4. *(a) Suppose that G is a CD* game. Then, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1 which is order independent. (b) Under the strict dominance relation, any compact and own-uppersemicontinuous game is a CD* game and, thus, the IESDS procedure defined in Definition 1 exists and is order independent.*

We have pointed out, in finite games, that the orginal definition of CD games might be conceptually little relevant to the property of order independence. We have thereby introduced a useful notion of CD* games for solving the problem of order dependence in infinite games. The definition of CD* not only captures Jackson's (1992) idea of "boundedness" that strategies are eliminated only by undominated strategies, but also it is immune from the problem of "discontinuity" at limit points as demonstrated in Examples 1 and 3. In CD* games, we have shown that the GKZ procedure is an order-independent iterated elimination procedure. The main driving force for order independence comes from the Hereditarity*/Monotonicity* property exhibited by a CD* game. While the class of CD* games is exclusive of all problematic games with the problem of order dependence, it abounds under the strict dominance, including all compact and own-uppersemicontinuous games. Consequently, the IESDS procedure in Definition 1 is always well-defined and order independent in COUSC games. (We would like to mention that Chen et al. (2007) presented an alternative definition of IESDS* by setting $c(X) = X \backslash DOM^S(X)$ and showed that IESDS* is well-defined and order independent in infinite games.)

4 Concluding remarks

In infinite games, Dufwenberg and Stegeman (2002) pointed out that the notion of IESDS might be an ill-behaved order-dependent procedure, even in the class of CD games where any dominated strategy has an undominated dominator, and they concluded that the problems of IESDS in infinite games are deeper than the possible nonexistence of the "best" dominating strategy. One major focus of this paper is to study various (transfinitely) iterated elimination procedures in the infinite case. We have shown a general existence of iterated elimination procedure. Following Gilboa et al.'s (1990) pioneering work, we have identified a fairly weak condition of Monotonicity* for the order independence on an abstract reduction system, which is closely related to the Aizerman property used in the choice-theoretic literature.

We have demonstrated that our approach is applicable to any form of iterated elimination processes in arbitrary strategic games. In addition, we have provided a full characterization of Monotonicity* by Hereditarity* in (in)finite games. We have also introduced a useful and variant notion of CD* games, including all compact and own-uppersemicontinuous (COUSC) games, and shown that the GKZ procedure is an order-independent iterated elimination procedure in CD* games. In particular, the IESDS procedure in Definition 1 is always well-defined and order independent in COUSC games.

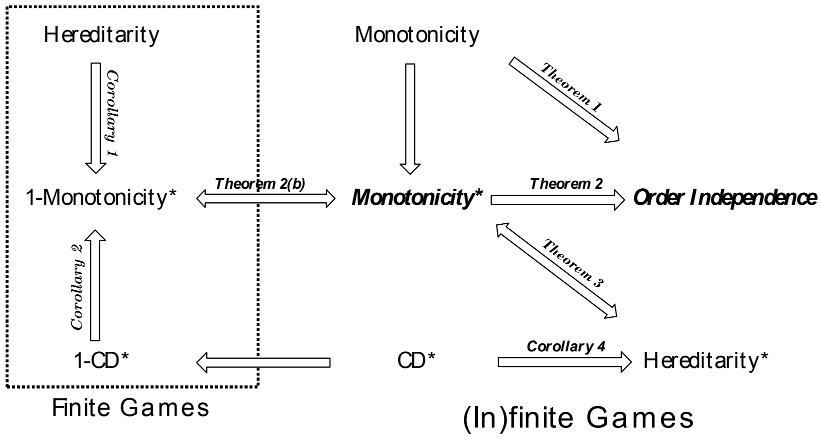

Table: Relationship between different conditions for order independence

We would like to point out that, except for the ZF axioms, the existence of iterated elimination procedure does not require the Axiom of Choice or the Well-Ordering Principle; this result improves the previous existence results of iterated elimination procedure in infinite games (e.g., Arieli (2012), Ritzberger (2002), Dufwenberg and Stegeman (2002) and Chen et al. (2007, 2015)). Our analysis of strategic games is completely topology-free and with no measure-theoretic assumption on the structure of the game, and it is applicable to any kind of iterated dominance in arbitrary games. Our framework in this paper can also be used to analyze the order independence of various forms of iterated elimination procedures in mixed extensions of finite games or general preference models used in game theory (cf. Chen et al. (2015)). Alternatively, we can define $c(X) = X \backslash DOM^S(X)$, which consists of all the elements in X that are undominated by any element in S. Our analysis in this paper is applicable to this alternative definition.

To close this paper, we would like to point out some possible extensions of this paper for future research. In this paper, we consider the order independence in terms of strategy profiles resulting from iterated elimination processes for games. Several papers discuss a slight variant of order independence in terms of "payoff outcomes" in finite strategic games (see, e.g., Marx and Swinkels (1997)) or in terms of "outcomes of play" in finite extensive games (see, e.g., Chen and Micali (2013) and Heifetz and Perea (2015)). The extension of this paper to such a variant of order independence in infinite games is an important subject for further research; Appendix II is an attempt in this direction. As we have emphasized, in this paper we focus on the existence and order independence of iterated elimination procedure. The exploration of iterated elimination procedures from an epistemic perspective is also an intriguing topic worth further investigation (see, e.g., Brandenburger et al. (2008)). Finally, Monotonicity* is not necessary for order independence (see Example 4 in Appendix I) and it is certainly interesting to explore the necessary and sufficient condition for order independence of iterated elimination procedure in general situations.

5 Appendix I: Proofs & Example 4

Proof of Theorem 1. (a) By transfinite recursion (see, e.g., Jech 2003, p.22), we define a sequence $\{X^\lambda\}_{\lambda \in Ord}$ (where Ord is the class of all ordinals) by

$$X^0 = S,\; X^{\lambda+1} = c\left(X^\lambda\right), \text{ and } X^\lambda = \cap_{\lambda'<\lambda} X^{\lambda'} \text{ for a limit ordinal } \lambda. \tag{1}$$

By the Axiom Schema of Separation (see, e.g., Jech 2003, p.7), $\{X^\lambda : \lambda \in Ord\}$ is a set because it is a subclass of the power set of S. Suppose, to the contrary, that $X^\lambda \neq X^{\lambda'}$ for any $\lambda \neq \lambda'$, then there is a bijection from $\{X^\lambda : \lambda \in Ord\}$ to Ord. By the Axiom Schema of Replacement (see, e.g., Jech 2003, p.13), Ord is a set, contradicting the fact that Ord is not a set. By (1), it follows that $X^\Lambda = X^{\Lambda+1} = c\left(X^\Lambda\right)$ for some $\Lambda \in Ord$. Let $\Lambda^0 = \inf\left\{\Lambda \in Ord : X^\Lambda = X^{\Lambda+1} = c\left(X^\Lambda\right)\right\}$. Then the sequence $\{X^\lambda\}_{\lambda \leq \Lambda^0}$ is a fast IEP on $(S, \rightarrow)$.

(b) Let $Z = c(Z)$. Obviously, $Z \subseteq X^0$. Assume, by induction, that $Z \subseteq X^{\lambda'}$ for all $\lambda' < \lambda$. By monotonicity, $c(Z) \subseteq c\left(X^{\lambda'}\right)$ for all $\lambda' < \lambda$. Therefore, $Z = c(Z) \subseteq X^\lambda$. That is, $Z \subseteq X^\lambda$ for all $\lambda \leq \Lambda$. Therefore, $X^\Lambda \supseteq \cup_{Z=c(Z)} Z$. Since $X^\Lambda = c\left(X^\Lambda\right)$, $X^\Lambda = \cup_{Z=c(Z)} Z$. □

To prove Theorem 2, we need the following three lemmas.

Lemma 1. *If $S \rightarrow^* X$ and $S \rightarrow^* Y$ imply that there exists T such that $X \rightarrow^* T$ and $Y \rightarrow^* T$, then the iterated elimination procedure is order independent.*

Proof. Assume by absurdity that there are two IEPs: $S \rightarrow^* X = c(X)$ and $S \rightarrow^* Y = c(Y)$, but $X \neq Y$. Then there exists T such that $X \rightarrow^* T$ and $Y \rightarrow^* T$. Therefore, $X = T = Y$. A contradiction. □

Lemma 2. *If c satisfies Monotonicity*, $S \rightarrow^* X \rightarrow Y$ implies $Y \rightarrow c(X)$.*

Proof. Let $S \rightarrow^* X \rightarrow Y$. Since c satisfies Monotonicity*, $c(Y) \subseteq c(X)$. Since $X \rightarrow Y$, $c(Y) \subseteq c(X) \subseteq Y \subseteq X$. By the definition of $\rightarrow$, $Y \rightarrow c(X)$. □

Lemma 3. *Suppose $S \rightarrow^* X$ via a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$. Then $c(X) \subseteq \cap_{\lambda<\Lambda} c\left(X^\lambda\right)$ if c satisfies Monotonicity*.*

Proof. Since c satisfies Monotonicity*, $c(X) \subseteq c\left(X^\lambda\right)$ for all $\lambda < \Lambda$. Therefore, $c(X) \subseteq \cap_{\lambda<\Lambda} c\left(X^\lambda\right)$. □

Proof of Theorem 2.[14] (a) Let $S \rightarrow^* X$ via a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$ and $S \rightarrow^* Y$ via a reduction sequence $\{Y^\lambda\}_{\lambda \leq \Lambda}$. We say the "*diamond property holds (for $\{X^\lambda\}_{\lambda \leq \Lambda}$ and $\{Y^\lambda\}_{\lambda \leq \Lambda}$)*" if there exists an $\Lambda \times \Lambda$-diamond grid $\{S^{\alpha\beta}\}_{\alpha \leq \Lambda;\, \beta \leq \Lambda}$ such that

1. for all $\lambda \leq \Lambda$, $S^{\lambda 0} = X^\lambda$ and $S^{0\lambda} = Y^\lambda$;
2. for all $\alpha, \beta \leq \Lambda$, $\{S^{\alpha\lambda}\}_{\lambda \leq \Lambda}$ and $\{S^{\lambda\beta}\}_{\lambda \leq \Lambda}$are reduction sequences.

[14] Our proof is inspired by Gilboa et al.'s (1990) idea for order independence of finite reduction sequences. We show that, under Monotonicity*, any pair of transfinite reduction sequences has a nice "diamond" property which gives rise to the order independence result.

That is, the diamond structure spreads over a grid of $\Lambda \times \Lambda$ fractals (cf. Figure 1).

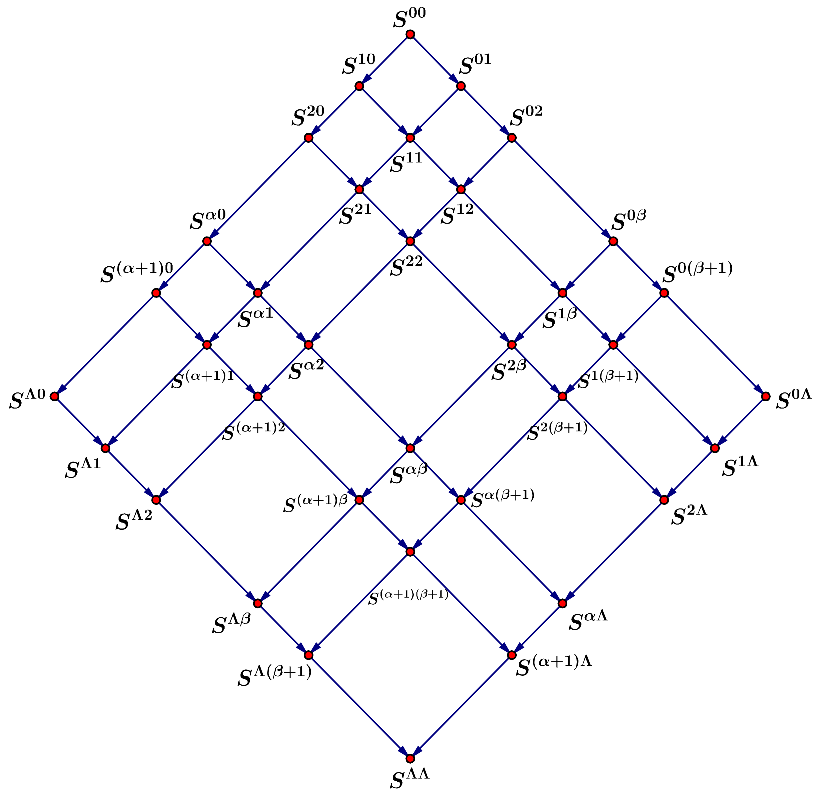

Fig. 1

Observe that: $S \to^* X$ and $S \to^* Y$ iff there exists an ordinal Λ such that $S \to^* X$ via a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$ and $S \to^* Y$ via a reduction sequence $\{Y^\lambda\}_{\lambda \leq \Lambda}$. By Lemma 1, it suffices to show that the diamond property holds true. We show it by (transfinite) induction on Λ. If $\Lambda = 1$, then $S \to X$ and $S \to Y$. By Lemma 2, $X \to c(S)$ and $Y \to c(S)$. Now assume that the diamond property holds for all $\lambda < \Lambda$. We distinguish two cases.

Case 1: Λ is a limit ordinal. Define $S^{\Lambda 0} \equiv X^\Lambda$ and $S^{\Lambda\beta} \equiv \cap_{\alpha<\Lambda} S^{\alpha\beta}$ for all $\beta < \Lambda$ and $\beta \neq 0$. Since $X^\Lambda = \cap_{\lambda<\Lambda} X^\lambda$, $S^{\Lambda 0} = \cap_{\alpha<\Lambda} S^{\alpha 0}$. By the induction hypothesis, for all $\beta < \Lambda$, we have

$$\begin{aligned}\left[S^{\alpha\beta} \to S^{\alpha(\beta+1)}\ \forall\alpha < \Lambda\right] &\Leftrightarrow \left[c\left(S^{\alpha\beta}\right) \subseteq S^{\alpha(\beta+1)} \subseteq S^{\alpha\beta}\ \ \forall\alpha < \Lambda\right] \\ &\Rightarrow \left[\cap_{\alpha<\Lambda} c\left(S^{\alpha\beta}\right) \subseteq \cap_{\alpha<\Lambda} S^{\alpha(\beta+1)} \subseteq \cap_{\alpha<\Lambda} S^{\alpha\beta}\right] \\ &\Leftrightarrow \left[\cap_{\alpha<\Lambda} c\left(S^{\alpha\beta}\right) \subseteq S^{\Lambda(\beta+1)} \subseteq S^{\Lambda\beta}\right].\end{aligned}$$

By Lemma 3, $c\left(S^{\Lambda\beta}\right) \subseteq \cap_{\alpha<\Lambda} c\left(S^{\alpha\beta}\right) \subseteq S^{\Lambda(\beta+1)} \subseteq S^{\Lambda\beta}$. Therefore, $S^{\Lambda\beta} \to S^{\Lambda(\beta+1)}$ for all $\beta < \Lambda$. (If β is a limit ordinal, $S^{\Lambda\beta} = \cap_{\alpha<\Lambda} S^{\alpha\beta} = \cap_{\alpha<\Lambda} \cap_{\beta'<\beta} S^{\alpha\beta'} = \cap_{\beta'<\beta} \cap_{\alpha<\Lambda} S^{\alpha\beta'} = \cap_{\beta'<\beta} S^{\Lambda\beta'}$.) Define $S^{\Lambda\Lambda} \equiv \cap_{\beta<\Lambda} S^{\Lambda\beta} = \cap_{\beta<\Lambda} \cap_{\alpha<\Lambda} S^{\alpha\beta}$. We find a reduction sequence

$\left\{S^{\Lambda\beta}\right\}_{\beta\leq\Lambda}$. Similarly, we find a reduction sequence $\left\{S^{\alpha\Lambda}\right\}_{\alpha\leq\Lambda}$ with $S^{\Lambda\Lambda} = \cap_{\alpha<\Lambda}\cap_{\beta<\Lambda}S^{\alpha\beta} = \cap_{\alpha<\Lambda}S^{\alpha\Lambda}$.

Case 2: Λ is a successor ordinal. By the induction hypothesis, there exists $(\Lambda-1)\times(\Lambda-1)$-diamond grid $\left\{S^{\alpha\beta}\right\}_{\alpha\leq\Lambda-1;\ \beta\leq\Lambda-1}$ for $\left\{X^{\lambda}\right\}_{\lambda\leq\Lambda-1}$ and $\left\{Y^{\lambda}\right\}_{\lambda\leq\Lambda-1}$. Define $S^{\Lambda 0}\equiv X^{\Lambda}$ and $S^{\Lambda(\beta+1)}\equiv c\left(S^{(\Lambda-1)\beta}\right)$ (and $S^{\Lambda\beta}\equiv\cap_{\beta'<\beta}S^{\Lambda\beta'}$ if β is a limit ordinal) for all $\beta\leq\Lambda-1$. Since $X^{\Lambda-1}\to X^{\Lambda}$, by the induction hypothesis, $S\to^{*}S^{(\Lambda-1)0}\to S^{\Lambda 0}$ and $S\to^{*}S^{(\Lambda-1)0}\to S^{(\Lambda-1)1}$. By Lemma 2, $S^{\Lambda 0}\to S^{\Lambda 1}$ and $S^{(\Lambda-1)1}\to S^{\Lambda 1}$. Again by induction on $\beta\leq\Lambda-1$, we have $S^{\Lambda\beta}\to S^{\Lambda(\beta+1)}$ for all $\beta\leq\Lambda-1$ and $S^{(\Lambda-1)\beta}\to S^{\Lambda\beta}$ for any $\beta\leq\Lambda-1$ (if β is a limit ordinal, the proof is totally similar to Case 1). Therefore, $\left\{S^{\Lambda\beta}\right\}_{\beta\leq\Lambda}$ and $\left\{S^{\alpha\beta}\right\}_{\alpha\leq\Lambda}$ for any $\beta\leq\Lambda-1$ are reduction sequences. Similarly, we can find a reduction sequence $\left\{S^{\alpha\Lambda}\right\}_{\alpha\leq\Lambda}$ such that $\left\{S^{\alpha\beta}\right\}_{\beta\leq\Lambda}$ for any $\alpha\leq\Lambda-1$ is a reduction sequence. That is, there exists an $\Lambda\times\Lambda$-diamond grid $\left\{S^{\alpha\beta}\right\}_{\alpha\leq\Lambda;\ \beta\leq\Lambda}$ for $\left\{X^{\lambda}\right\}_{\lambda\leq\Lambda}$ and $\left\{Y^{\lambda}\right\}_{\lambda\leq\Lambda}$. Therefore, the diamond property holds.

(b) Consider $S\to^{*}X\to^{*}Y$. By Theorem 2(a), it suffices to check $c(Y)\subseteq c(X)$. Since S is a finite set, there exists a natural number N such that $X\to^{*}Y$ via a finite reduction sequence $\{X^{n}\}_{n\leq N}$ (with $X^{n-1}\neq X^{n}$ for all $n\leq N$). By 1-Monotonicity*, $c(X^{1})\subseteq c(X^{0})=c(X)$. Assume inductively that $c(X^{n-1})\subseteq c(X)$ for all $n\leq N$. Since $S\to^{*}X^{n-1}\to X^{n}$, $c(X^{n})\subseteq c(X^{n-1})\subseteq c(X)$ by 1-Monotonicity*. Thus, $c(Y)=c\left(X^{N}\right)\subseteq c(X)$. □

Proof of Corollary 1. Suppose that $X\to Y$. That is, $c(X)\subseteq Y\subseteq X$. By Hereditarity, we have

$$\begin{aligned}[(Y\cap DOM(X))\subseteq DOM(Y)] &\Leftrightarrow [Y\backslash(Y\cap DOM(X))\supseteq Y\backslash DOM(Y)]\\ &\Leftrightarrow [Y\backslash DOM(X)\supseteq Y\backslash DOM(Y)]\\ &\Rightarrow [X\backslash DOM(X)\supseteq Y\backslash DOM(Y)].\end{aligned}$$

That is, $c(Y)\subseteq c(X)$ if $X\to Y$. By Theorem 2(b), the iterated elimination procedure for G is order-independent. □

Proof of Corollary 2. Suppose that $S\to^{*}X\to Y$. Since $X\to Y$, $c(X)\subseteq Y\subseteq X$. Since G is a 1-CD* game, $Y\cap DOM(X)\subseteq DOM^{c(Y)}(Y)\subseteq DOM(Y)$. By the proof of Corollary 1, we obtain that $S\to^{*}X\to Y$ implies $c(Y)\subseteq c(X)$. By Theorem 2(b), the iterated elimination procedure for G is order independent. □

Proof of Corollary 3. Suppose that $S\to^{*}X\to Y$. Then $c(X)\subseteq Y\subseteq X$. Since G is 1-CD* and $S\to^{*}X\to X$, $DOM(X)=DOM^{c(X)}(X)$. Thus, $DOM(X)=DOM^{Y}(X)=DOM^{c(X)}(X)$. Therefore, we obtain

$$\begin{aligned}[X\to Y] &\Leftrightarrow [Y\subseteq X \text{ and } X\backslash Y\subseteq DOM(X)]\\ &\Leftrightarrow \left[Y\subseteq X \text{ and } X\backslash Y\subseteq DOM^{Y}(X)\right]\\ &\Leftrightarrow \left[X\to^{\mathrm{GKZ}}Y\right].\end{aligned}$$

That is, for any finite 1-CD* game, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1. By Corollary 2, the GKZ procedure is order independent. □

Lemma 4. *The choice rule c for c-rationalizability satisfies 1-Monotonicity*.*
Proof of Lemma 4. Let $X \searrow_J Z$ denote "supported restriction Z by coalition J given X". Consider $X \to Y$. Then $X \supseteq Y \supseteq c(X) = \cap_{Z \in \mathcal{F}(X)} Z \neq \varnothing$ by Ambrus's (2006) Proposition 1. Since $Y \cap Z \supseteq Y \cap c(X) \neq \varnothing$ for $Z \in \mathcal{F}(X)$, by Ambrus's (2006) Lemmas 1 and 2, $Y \searrow_J (Y \cap Z)$. Then $Y \cap Z \in \mathcal{F}(Y)$ for all Z in $\mathcal{F}(X)$. Thus, $c(Y) = \cap_{Z \in \mathcal{F}(Y)} Z \subseteq \cap_{Z \in \mathcal{F}(X)} (Y \cap Z) \subseteq \cap_{Z \in \mathcal{F}(X)} Z = c(X)$. □

Proof of Theorem 3. Suppose $S \to^* X \to^* Y$. Then $Y \subseteq X$. Thus, we have

$$\begin{aligned}
[(Y \cap DOM(X)) \subseteq DOM(Y)] &\Leftrightarrow [Y \backslash (Y \cap DOM(X)) \supseteq Y \backslash DOM(Y)] \\
&\Leftrightarrow [Y \backslash DOM(X) \supseteq Y \backslash DOM(Y)] \\
&\Leftrightarrow [X \backslash DOM(X) \supseteq Y \backslash DOM(Y)].
\end{aligned}$$

That is, $(Y \cap DOM(X)) \subseteq DOM(Y)$ iff $c(X) \supseteq c(Y)$. Therefore, Hereditarity* and Monotonicity* are equivalent. By Theorem 2(a), Hereditarity* implies that the iterated elimination procedure is order independent. □

Proof of Corollary 4. (a) Since any CD* game is 1-CD*, by Corollary 3, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1. Suppose $S \to^* X \to^* Y$. Since G is a CD* game, $(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y) \subseteq DOM(Y)$. That is, Hereditarity* holds. By Theorem 3, the GKZ procedure is order independent.

(b) Suppose that $S \to^* X \to^* Y$ via a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$. Let $y \succ_X x$ for some $y \in X$ and $x \in Y$. Then, $\exists i \in N$ such that $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. Since G is a COUSC game, by the proof of Dufwenberg and Stegeman's (2002) Lemma, $\exists z^* \in S$ such that, for all $y' \in Y$, (i) $u_i\left(z_i^*, y'_{-i}\right) > u_i\left(x_i, y'_{-i}\right)$ and (ii) $u_j(z_j^*, x_{-j}) \geq u_j(s_j, x_{-j})$ for all $j \in N$ and all $s_j \in S_j$. Since $x \in Y \subseteq X^\lambda$, $z^* \in X^\lambda$ for all $\lambda < \Lambda$. Thus, $z^* \in \cap_{\lambda < \Lambda} X^\lambda = Y$. By (i) and (ii), $z^* \succ_Y x$ and $z^* \in c(Y)$. Therefore, $(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y)$, i.e., G is a CD* game. By Theorem 1(a) and Corollary 4(a), the IESDS procedure defined in Definition 1 exists and is order independent in COUSC games. □

Example 4 [Monotonicity* is not necessary for order independence]. Consider $S = \{x, y, z\}$ with the following choice function c (we abuse notation by writing, for example, xy instead of $\{x, y\}$):

$X \subseteq S$	xyz	xy	xz	yz	x	y	z
$c(X)$	x	x	z	z	$\varnothing$	$\varnothing$	$\varnothing$

Any IEP leads to $\varnothing$. But, $S \to xz$ fails to imply $c(xz) \subseteq c(S)$.

6 Appendix II: NWD

In this appendix, we offer an alternative proof of Marx and Swinkels's (1997) result on the "outcome" order independence for nice weak dominance (NWD).

Consider a finite game $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$. Let $X = \times_{i\in N} X_i \subseteq S$. A *snapshot of* X is defined as a game $(N, \{\Sigma_i\}_{i\in N}, \{v_i\}_{i\in N})$ such that, for all $i \in N$, there are surjections $e_i : X_i \mapsto \Sigma_i$ satisfying for all $\sigma \in \Sigma$, $v(\sigma) = u(x)$ $\forall x \in e^{-1}(\sigma)$, where $\Sigma = \times_{i\in N}\Sigma_i$, $e(x) = (e_i(x_i))_{i\in N}$ and $e^{-1}(\sigma) = \{x \in X : e(x) = \sigma\}$. Denote $X \sim Y$ if they have a common snapshot.

Claim NWD. *For any finite game G, iterated elimination of nicely weakly dominated strategies are outcome order independent – i.e., if $\{X^\lambda\}_{\lambda\leq\Lambda}$ and $\{Y^\lambda\}_{\lambda\leq\Lambda'}$ are two finite IEPs of product sets of S, then $X^\Lambda \sim Y^{\Lambda'}$.*

Lemma NWD. *(a) $X \sim X' \Rightarrow c(X) \sim c(X')$. (b) $X \to Y \Rightarrow Y \to Z$ for some $Z \sim c(X)$ (and $Z \supseteq c(Y)$). (c) Let $[X] \to [Y]$ denote "$X' \to Y'$ for some $X' \sim X$ and $Y' \sim Y$". Then $[X] \to [Y] \Rightarrow [Y] \to [c(X)]$.*

Proof of Lemma NWD. (a) Suppose that $(N, \{\Sigma_i\}_{i\in N}, \{v_i\}_{i\in N})$ is a common snapshot of X and X' (via the corresponding surjections e and e'). Then, for all $\sigma \in \Sigma$, $e_i^{-1}(\sigma_i) \subseteq DOM_i(X) \Leftrightarrow \sigma_i \in DOM_i(\Sigma) \Leftrightarrow e_i'^{-1}(\sigma_i) \subseteq DOM_i(X')$ $\forall i \in N$. Thus, for all $i \in N$, $c_i(X) = \cup_{\sigma_i\in c_i(\Sigma)} e_i^{-1}(\sigma_i)$ and $c_i(X') = \cup_{\sigma_i\in c_i(\Sigma)} e_i'^{-1}(\sigma_i)$. Therefore, $c(X)$ and $c(X')$ have a common snapshot $(N, \{c_i(\Sigma)\}_{i\in N}, \{v_i\}_{i\in N})$ by the (restricted) surjections $e|_{c(X)}$ and $e'|_{c(X')}$, respectively. That is, $c(X) \sim c(X')$.

(b) Let $Z = \times_{i\in N}(c_i(X) \cup c_i(Y))$. Since $X \to Y$, $c(X) \subseteq Y \subseteq X$. So $c(Y) \subseteq Z \subseteq Y$, i.e., $Y \to Z$. It remains to show $Z \sim c(X)$. Let $z_i \in c_i(Y) \backslash c_i(X)$. Then, z_i is NWD on X by some $\widetilde{z}_i \in X_i$ and $u(z_i, y_{-i}) = u(\widetilde{z}_i, y_{-i})$ $\forall y_{-i} \in Y_{-i} \subseteq X_{-i}$. Since $Z \subseteq Y$, $u(z_i, z_{-i}) = u(\widetilde{z}_i, z_{-i})$ $\forall z_{-i} \in Z_{-i}$. By finiteness of X and transitivity of NWD, there exists $\widetilde{z}_i \in c_i(X)$ such that $u(z_i, z_{-i}) = u(\widetilde{z}_i, z_{-i})$ $\forall z_{-i} \in Z_{-i}$. Therefore, Z has a snapshot $(N, \{c_i(X)\}_{i\in N}, \{u_i\}_{i\in N})$ thorough the surjection $\widetilde{e}_i : Z_i \mapsto c_i(X)$ such that

$$\widetilde{e}_i(z_i) = \begin{cases} z_i, & \text{if } z_i \in c_i(X) \\ \widetilde{z}_i, & \text{if } z_i \in c_i(Y)\backslash c_i(X) \end{cases}.$$

Hence, $Z \sim c(X)$. By construction of Z, $c(Y) \subseteq Z \sim c(X)$.

(c) Suppose $[X] \to [Y]$. Then, there exist $X' \sim X$ and $Y' \sim Y$ such that $X' \to Y'$. By (b), $Y' \to Z'$ for some $Z' \sim c(X')$. By (a), $c(X) \sim c(X')$. Therefore, $[Y] \to [c(X)]$. □

Proof of Claim NWD. Without loss of generality, assume that $\{X^\lambda\}_{\lambda\leq\Lambda}$ and $\{Y^\lambda\}_{\lambda\leq\Lambda}$ be two finite IEPs of product sets of S. Then $[X^\lambda] \to [X^{\lambda+1}]$ and $[Y^\lambda] \to [Y^{\lambda+1}]$ for all $\lambda < \Lambda$. Similarly to the proof of Theorem 2(a), we can show, by Lemma NWD(c), that $\Lambda \times \Lambda$-grid $\{[S^{\alpha\beta}]\}_{\alpha\leq\Lambda;\beta\leq\Lambda}$ holds and $[X^\Lambda] = [S^{\Lambda\Lambda}] = [Y^\Lambda]$. □

References

1. Ambrus, A.: Coalitional rationalizability. QJE 121(2006), 903-929.

2. Ambrus, A.: Theories of coalitional rationality. JET 144(2009), 676-695.

3. Apt, K.R.: Uniform proofs of order independence for various strategy elimination procedures. B.E. J. Theoretical Econ. 4, Article 5 (2004).

4. Apt, K.R.: Direct proofs of order independence. Econ. Bull. 31(2011), 106-115.

5. Arieli, I.: Rationalizability in continuous games. JME 46(2012), 912-924.

6. Asheim, G.B., Dufwenberg, M.: Admissibility and common belief. GEB 42(2003), 208-234.

7. Bernheim, B.D.: Rationalizable strategic behavior. Econometrica 52(1984), 1007-1028.

8. Borgers, T.: Pure strategy dominance, Econometrica, 61(1993), 423-430.

9. Brandenburger, A., Friedenberg, A., Keisler, H.J.: Admissibility in games. Econometrica 76(2008), 307-352.

10. Chen, J., Micali, S.: The order independence of iterated dominance in extensive games. TE 8(2013), 125-163.

11. Chen, Y.C., Long, N.V., Luo, X.: Iterated strict dominance in general games. GEB 61(2007), 299-315.

12. Chen, Y.C., Luo, X., Qu, C.: Rationalizability in general situations. ET, forthcoming (2015).

13. Cubitt, R.P., Sugden, R.: The reasoning-based expected utility procedure. GEB 71(2011), 328-338.

14. Dekel, E., Fudenberg, D.: Rational behavior with payoff uncertainty. JET 52(1990), 243-267.

15. Dufwenberg, M., Stegeman, M.: Existence and uniqueness of maximal reductions under iterated strict dominance. Econometrica 70(2002), 2007-2023.

16. Duggan, J., Le Breton, M.: Choice-theoretic solutions for strategic form games. Mimeo, University of Rochester (2014).

17. Gilboa, I., Kalai, E., Zemel, E.: On the order of eliminating dominated strategies. OR Lett. 9(1990), 85-89.

18. Green, E.: Iterated elimination of dominated strategies in countable-strategy games. Mimeo, Penn State University (2011).

19. Gul, F.: Rationality and coherent theories of strategic behavior. JET 70(1996), 1-31.

20. Halpern, J.Y., Pass, R.: Iterated regret minimization: a new solution concept. GEB 74(2012), 184-207.

21. Heifetz, A., Perea, A.: On the outcome equivalence of backward induction and extensive form rationalizability. IJGT 44(2015), 37-59.

22. Hillas, J., D. Samet, D.: Weak dominance: a mystery cracked. Mimeo, Tel Aviv University (2014).

23. Jackson, M.: Implementation in undominated strategies: a look at bounded mechanisms. RES 59(1992), 757-775.

24. Jech, T.: Set Theory. Springer-Verlag, Berlin (2003).

25. Lipman, B.L.: A note on the implication of common knowledge of rationality. GEB 6(1994), 114-129.

26. Luo, X.: General systems and φ-stable sets – a formal analysis of socioeconomic environments. JME 36(2001), 95-109.

27. Luo, X., C.C. Yang: Bayesian coalitional rationalizability. JET 144(2009), 248-263.

28. Marx, L.M., Swinkels, J.M.: Order independence for iterated weak dominance. GEB 18(1997), 219-245.

29. Milgrom, P., Roberts, J.: Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58(1990), 1255-1278.

30. Moulin, H.: Choice functions over a finite set: a summary. SCW 2(1985), 147-160.

31. Newman, M.H.A.: On theories with a combinatorial definition of "equivalence". Annals of Math. 43(1942), 223-243.

32. Osborne, M.J., Rubinstein, A.: A Course in Game Theory. The MIT Press, MA (1994).

33. Oyama, D., Tercieux, O.: Iterated potential and the robustness of equilibria. JET 144(2009), 1726-1769.

34. Pearce, D.G.: Rationalizable strategic behavior and the problem of perfection. Econometrica 52(1984), 1029-1051.

35. Ritzberger, K.: Foundations of Non-Cooperative Game Theory. Oxford University Press, Oxford (2002).

36. Sen, A.: Internal consistency of choice. Econometrica 61(1993), 495-521.

37. Tercieux, O.: p-best response set. JET 131(2006), 45-70.

38. Trost, M.: An epistemic rationale for order independence. IGTR 16(2014), 1440002.